Quantum Fluctuations of Coherent Light in Nonlinear Media

July 9, 2002

This work is dedicated to Dr. Arani Chakravarti & Dr. Samita Sil without whose encouragement it would be impossible for me to complete the work.

ACKNOWLEDGEMENT

For a Ph.D work the essential condition is that there will be a student and a supervisor. In the present case the condition is fulfilled by me (student) and Dr. Swapan Mandal (my supervisor). I am thankful to my supervisor for his help and guidance. The essential condition stated above is not sufficient for a PhD work and one needs a lot of journals, preprints, reprints, software and scientific discussions in order to achieve the sufficiency condition. In last four years many people have helped me to achieve the sufficiency condition. I am grateful to all of them. It is not possible to acknowledge everyone by name. Still I would like to express my sincere thanks to Professor E C G Sudarshan, Professor B K Talukdar, Professor F M Fernández, Dr. Sreekantha Sil and Dr. Arani Chakravarti for some valuable technical discussions. I also wish to thank Chiranjib Sur, Madhumita Gupta, Anuj Kumar Saw, Susanta Madhab Das, Pika Jha, Purnendu Chakraborty, Abhijit Sen, Tarun Kanti Ghosh, Tanya Bhattacharya, Sutapa Datta, Moumita Maiti, Bipul Sarkar and Biswajit Sen for their cooperation and technical support.

I am also indebted to Kishore Mandal and other nonteaching staff of our department for their constant cooperation.

I am grateful to the Council of Scientific and Industrial Research, India for their financial support during my research period.

I would like to take this opportunity to express my deep sense of gratitude to my grandmother, parents, uncles, aunts, sisters, brothers and all other family members who were always there with me with their constant encouragement and support. Lastly, but certainly not the least, I wish to thank my friend Papia Chowdhury for her cooperation and interest in the present work without which it would not be possible for me to devote all my time to the present study.

Date: 6^{th} May, 2002 (Anirban Pathak)

ABSTRACT

There are substantial quantum fluctuations even in a pure coherent state. For example, the quadrature fluctuation in coherent state $|\alpha>$ is $\frac{1}{2}$ and the fluctuation in photon number is $|\alpha|^2$. Further, randomization in the case of general classical states which are random superposition of coherent states can only increase these fluctuations. But there are some phenomena in which quantum fluctuations reduce below the coherent state level. For example, an electromagnetic field is said to be electrically squeezed field if uncertainties in the quadrature phase observable X is less than the coherent state level (i.e. $(\triangle X)^2 < \frac{1}{2}$). Correspondingly a magnetically squeezed field is one for which $(\triangle \dot{X})^2 < \frac{1}{2}$. On the other hand, antibunching is a phenomenon in which the fluctuation in photon number is reduced below the Poisson level (i.e $(\triangle N)^2 << N >$). All phenomena in which quantum fluctuations reduce below those of the coherent state level are called nonclassical. It can be noted that the interaction of dielectric medium with an intense electromagnetic field can gives rise to these nonclassical nonlinear optical effects. These effects which help us to understand the nature of quantum world in further detail are also important from the application point of view. Keeping that in mind, the present thesis is devoted to the study of quantum fluctuations of coherent light in non-linear media.

The interaction of coherent light with a nonlinear medium is modeled here by a general quantum anharmonic oscillator. The model of a quartic anharmonic oscillator emerges if the lowest order of nonlinearity (i.e third order nonlinear susceptibility) is assumed. The quartic oscillator model has considerable importance in the study of non-linear and quantum optical effects present in a nonlinear medium of inversion symmetry. Silica crystals constitute an inversion symmetric third order nonlinear medium and these crystals are used to construct optical fibers. In optical communications electromagnetic beams pass through these fibers and the interaction of the electromagnetic field (single mode) with the fiber can be described by the one dimensional quartic anharmonic oscillator Hamiltonian. Depending on the nature of nonlinearity in physical problems the treatment of higher anharmonic oscillators assumes significance.

Anharmonic oscillator models are not exactly solvable in a closed analytical form. But we need operator solutions of the equations of motion corresponding to these models in order to study the quantum fluctuations of coherent light in nonlinear media. So we have two alternatives, either we can use an approximate Hamiltonian which is exactly solvable or we can use an approximate operator solution. Operator solutions (solutions in the Heisenberg approach) of anharmonic oscillator problems were not available since the existing methods tend to introduce inordinate mathematical complications in a detailed study. Due to the unavailability of the operator solutions people were bound to use rotating wave approximated Hamiltonians to study the quantum fluctuations of coherent light in nonlinear media. The situation has improved in the recent past and many solutions of anharmonic oscillators have appeared. Some of these solutions are obtained as a part of the present work. For example, we have constructed a second order operator solution for quartic oscillator and have generalized all first order operator solutions available for the quartic oscillator to the m-th anharmonic oscillator. From the generalized solutions we observe that there exists an apparent discrepancy between the solutions obtained by different techniques. Then the question arises: Which solution should be used for physical applications. Therefore, we compare different solutions and conclude that all correct solutions are equivalent and the apparent discrepancy is due to the use of different ordering of the operators.

We use these solutions to investigate the possibilities of observing different optical phenomena in a nonlinear dielectric medium. To be precise, we have studied quantum phase fluctuations

of coherent light in third order inversion symmetric nonlinear medium. Fluctuations in phase space quadrature for the same system are studied and the possibility of generating squeezed state is reported. Fluctuations in photon number are studied and the nonclassical phenomenon of antibunching is predicted. We have generalized the results obtained for third order nonlinear media and have studied the interaction of an intense laser beam with a general $(m-1)$-th order nonlinear medium. Aharonov Anandan nonadiabatic geometric phase is also discussed in the context of $(m-1)$-th order nonlinear medium.

LIST OF PUBLICATIONS ON SOME OF WHICH MAIN RESULTS OF THIS THESIS WERE PUBLISHED

1. Enhanced and reduced phase fluctuations of coherent light coupled to a quantum quartic anharmonic oscillator, Pathak A and Mandal S 1999 *Optics and Optoelectronics : Theory Devices and Application Vol I* Eds: NIjhawan O P, Gupta A K, Musala A K and Singh K (New Delhi: Narosa) 188.

2. Generalized quantum anharmonic oscillator using an operator ordering approach, Pathak A 2000 *J. Phys. A* **33** 5607-5613.

3. Phase fluctuations of coherent light coupled to a nonlinear medium of inversion symmetry, Pathak A and Mandal S 2000 *Phys. Lett. A.* **272** 346-352.

4. Quantum oscillator of quartic anharmonicity: second order solution, Pathak A and Mandal S 2001 *Phys. Lett. A* **286** 261-276.

5. Aharonov Anandan phase for the quasi exactly solvable Bose Systems, Pathak A 2001 *Proc. of Nat. Conf. on Laser and Its Application* 159.

6. Classical and quantum oscillators of sextic and octic anharmonicities, Pathak A and Mandal S 2002 *Phys. Lett. A* in press

7. [1]Spatio-temporal history of decay, Pathak A, submitted to *Phys. Rev. A.*

8. Perturbation theory free from secular terms for quantum mechanical anharmonic oscillators: Frequency operators Pathak A and Fernández F M, submitted to *Annals of Physics*.

9. Possibilities of observing Aharonov Anandan geometric phase for a generalized anharmonic oscillator, Pathak A, submitted to *Journal of Optics B: Quantum and Semiclassical Optics.*

[1]Not included in this thesis.

Contents

Chapter 1

Interaction of electromagnetic field with matter

Since the invention of laser people have observed different nonlinear optical effects in dielectric media interacting with an intense electromagnetic field. Studies into these effects are extremely important because some of them (such as squeezing and antibunching of photons) do not have any classical analogue. These purely quantum mechanical effects are also important from the application point of view. Keeping that in mind, in the present work we have discussed a class of physical systems in which different nonlinear and quantum optical effects may be observed. To be more specific, we consider a physical system in which an intense electromagnetic field is interacting with an $(m-1)$-th order nonlinear medium. The physical system is modeled by a generalized anharmonic oscillator (AHO). The operator solution of the equation of motion corresponding to our model Hamiltonian is developed using different techniques and is exploited to study the possibilities of observing higher harmonic generation, bunching and anti-bunching of photons, self induced transparency and squeezing. In addition to this, fluctuations in quantum phase of the output field is also studied with the knowledge of field operators. Geometric phase of the output field is also taken care of.

1.1 The model

The present thesis is devoted to the study of the interaction of a single-mode intense electromagnetic field with a nonlinear medium. In such a nonlinear interaction nonclassical effects like squeezing and antibunching of photons may be produced. In fact, here we study the possibilities of observing these nonclassical phenomena in the physical system of our interest. We can continue our study by using either a semiclassical theory or a quantum theory. Actually, a semiclassical theory, where the radiation is represented by a classical wave and the atom is quantized, can treat many problems in light-matter interaction. However, there are cases where a semiclassical theory turns out to be inadequate. These include spontaneous emission, Lamb shift, resonance fluorescence, the anomalous gyromagnetic moment of the electron and “nonclassical” states of light (e.g. squeezed states). So if we need to study the possibilities of observing nonclassical states then we have to use a quantum theory instead of a semiclassical one and as an essential requirement of the quantum theory we have to quantize the electromagnetic field. In the next subsection the electromagnetic field is quantized and the model Hamiltonian of our physical system is constructed in the subsequent subsection.

1.1.1 Quantization of the electromagnetic field

To quantize electromagnetic field let us recall some basic results of electrodynamics. The Maxwell equations describing the propagation of electromagnetic fields are

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial \mathbf{t}} \\ \nabla . \mathbf{D} &= \rho \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla . \mathbf{B} &= 0 \end{aligned} \tag{1.1}$$

where $\mathbf{J}$ is the current density; the other symbols have their conventional definitions. In a homogeneous isotropic medium $\mathbf{B}$ and $\mathbf{D}$ are related to $\mathbf{H}$ and $\mathbf{E}$ by

$$\begin{aligned} \mathbf{B} &= \mu \mathbf{H} \\ \mathbf{D} &= \epsilon \mathbf{E} \end{aligned} \tag{1.2}$$

where μ and ϵ are, respectively, the magnetic permeability and the dielectric constant of the medium. If we restrict ourselves to a charge free medium, then we have

$$\begin{aligned} \rho &= 0 \\ \mathbf{J} &= 0\,. \end{aligned} \tag{1.3}$$

Consider the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ inside a volume V bounded by a surface S of perfect conductivity. The tangential component of $\mathbf{E}$, $-\mathbf{n} \times \mathbf{n} \times \mathbf{E}$, and the normal component of $\mathbf{H}$, $\mathbf{n}.\mathbf{H}$ will both be zero on S ($\mathbf{n}$ is the unit vector normal to S). We can expand $\mathbf{E}$ and $\mathbf{H}$ in terms of two orthogonal sets of vector fields $\mathbf{E}_a$ and $\mathbf{H}_a$, respectively. These sets which were originally introduced by Slater [1] and nicely explained by Yariv [2] obey the relations

$$k_a \mathbf{E}_a = \nabla \times \mathbf{H}_a \tag{1.4}$$

and

$$k_a \mathbf{H}_a = \nabla \times \mathbf{E}_a \tag{1.5}$$

where k_a is to be considered, for the moment, a constant. The tangential component of $\mathbf{E}_a$ on S is zero, i.e

$$\mathbf{n} \times \mathbf{E}_a = 0 \, on \, S. \tag{1.6}$$

If we take the curl of both sides of the equations (1.4-1.5) and use the identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla \left(\nabla . \mathbf{A} \right) - \nabla^2 \mathbf{A} \tag{1.7}$$

then we obtain the familiar wave equations

$$\begin{aligned} \nabla^2 \mathbf{E}_a &= k_a^2 \mathbf{E}_a \\ \nabla^2 \mathbf{H}_a &= k_a^2 \mathbf{H}_a\,. \end{aligned} \tag{1.8}$$

Now we can write the total resonator fields of $\mathbf{E}$ and $\mathbf{H}$ as

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -\sum_a \frac{\omega_a}{\sqrt{\epsilon}} q_a(t) \mathbf{E_a}(r) \\ \mathbf{H}(\mathbf{r}, t) &= \sum_a \frac{1}{\sqrt{\mu}} p_a(t) \mathbf{H_a}(r) \end{aligned} \tag{1.9}$$

where $\omega_a = \frac{k_a}{\sqrt{\mu\epsilon}}$. $q_a(t)$ and $p_a(t)$ are measures of the field amplitude in a-th mode. Substituting (1.9) in the first two Maxwell equations (1.1) and using (1.4-1.5) we obtain

$$\begin{aligned} \dot{p}_a &= -\omega_a^2 q_a \\ \dot{q}_a &= p_a \,. \end{aligned} \tag{1.10}$$

From (1.10) we have

$$\begin{aligned} \ddot{q}_a + \omega_a^2 q_a &= 0 \\ \ddot{p}_a + \omega_a^2 p_a &= 0\,. \end{aligned} \tag{1.11}$$

This identifies $\omega_a = \frac{k_a}{\sqrt{\mu\epsilon}}$ as the radian oscillation frequency of the a-th mode.

From (1.9) we can easily show that

$$q_a(t) = -\frac{\sqrt{\epsilon}}{\omega_a}\int_V \mathbf{E}(\mathbf{r},t).\mathbf{E_a}(\mathbf{r})dv \tag{1.12}$$

and

$$p_a(t) = \sqrt{\mu}\int_V \mathbf{H}(\mathbf{r},t).\mathbf{H}_a(\mathbf{r})dv. \tag{1.13}$$

Now we conclude that the electromagnetic field can be specified either by $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ or, alternatively, by the dynamical variables $q_a(t)$ and $p_a(t)$. The total energy Hamiltonian is

$$H_0 = \frac{1}{2}\int_V (\mu\mathbf{H}.\mathbf{H} + \epsilon\mathbf{E}.\mathbf{E})dv \tag{1.14}$$

where the subscript zero is used to distinguish the free field Hamiltonian (H_0) from the total Hamiltonian of the system. Substituting the expansions of $\mathbf{E}$ and $\mathbf{H}$ (1.9) in (1.14) we obtain

$$H_0 = \sum_a \frac{1}{2}\left(p_a^2 + \omega_a^2 q_a\right). \tag{1.15}$$

This has the basic form of a sum of harmonic oscillator Hamiltonians. The dynamical variables p_a and q_a constitute canonically conjugate variables. This can be seen by considering Hamilton's equations of motion relating $\dot{p}_a$ to q_a and $\dot{q}_a$ to p_a. The equations are

$$\begin{aligned} \dot{p}_a &= -\frac{\partial H}{\partial q_a} &= -\omega_a^2 q_a \\ \dot{q}_a &= \frac{\partial H}{\partial p_a} &= p_a\,. \end{aligned} \tag{1.16}$$

These are formally identical with (1.11), obtained from Maxwell's equations.

The quantization of the electromagnetic field is finally achieved by considering p_a and q_a as formally equivalent to the momentum and coordinate of a quantum mechanical harmonic oscillator, thus taking the commutator relations connecting the dynamical variables as

$$\begin{aligned} [p_a, p_b] &= [q_a, q_b] = 0 \\ [q_a, p_b] &= i\delta_{a,b} \end{aligned} \tag{1.17}$$

where we have chosen to work in a system of units in which $\hbar = 1$. At this point we can introduce bosonic creation and annihilation operators as

$$\begin{aligned} a_l^\dagger(t) &= \left(\frac{1}{2\omega_l}\right)^{\frac{1}{2}}[\omega_l q_l(t) - ip_l(t)] \\ a_l(t) &= \left(\frac{1}{2\omega_l}\right)^{\frac{1}{2}}[\omega_l q_l(t) + ip_l(t)]\,. \end{aligned} \tag{1.18}$$

Solving (1.18) for p_l and q_l we obtain

$$\begin{aligned} p_l(t) &= i\left(\frac{\omega_l}{2}\right)^{\frac{1}{2}}\left[a_l^\dagger(t) - a_l(t)\right] \\ q_l(t) &= \left(\frac{1}{2\omega_l}\right)^{\frac{1}{2}}\left[a_l^\dagger(t) + a_l(t)\right]. \end{aligned} \tag{1.19}$$

Now we can express the free field Hamiltonian in terms of annihilation and creation operators as

$$H_0 = \sum_l \omega_l (a_l^\dagger a_l + \frac{1}{2}). \tag{1.20}$$

Thus we have quantized the electromagnetic field in vacuum. But the aim of the present thesis is to study the interaction of a single-mode intense electromagnetic field with a nonlinear medium. Therefore, the interaction part of the Hamiltonian should be taken into account.

1.1.1.1 Conventions used

Before we go into further detail let us state the conventions which are used in the present thesis.

i) We work in units in which $\hbar = 1$.

ii) The applied electromagnetic field is chosen to be a single mode electromagnetic field having unit frequency (i.e $\omega = 1$).

iii) Mass of the harmonic oscillator equivalent to the applied field is taken to be unity.

iv) Lower case $x(t)$ and $\dot{x}(t)$ are to be treated as time development of the position and momentum variables in the classical sense and the equivalent operator representation are to be made by using the corresponding upper case letters $X(t)$ and $\dot{X}(t)$. Thus

$$q(t) = X(t) = \frac{1}{\sqrt{2}}\left[a^\dagger + a\right], \tag{1.21}$$

$$p(t) = m\dot{q}(t) = m\dot{X}(t) = \dot{X}(t) = \frac{i}{\sqrt{2}}\left[a^\dagger - a\right], \tag{1.22}$$

(since $m = 1$) and

$$[X(t), \dot{X}(t)] = i. \tag{1.23}$$

1.1.2 Construction of the model Hamiltonian

With the above conventions the free field Hamiltonian (1.20) of our system reduces to

$$\begin{aligned} H_0 &= \frac{X^2}{2} + \frac{\dot{X}^2}{2} \\ &= a^\dagger a + \frac{1}{2}. \end{aligned} \tag{1.24}$$

Again from (1.9) we observe that

$$\mathbf{E} \propto q \propto \left(a^\dagger + a\right) \tag{1.25}$$

and

$$\mathbf{B} \propto p \propto \left(a^\dagger - a\right). \tag{1.26}$$

Now an intense electromagnetic field interacting with a dielectric medium induces a macroscopic polarization $(\mathbf{P})$ having a general form

$$\mathbf{P} = \chi_1\mathbf{E} + \chi_2\mathbf{E}\mathbf{E} + \chi_3\mathbf{E}\mathbf{E}\mathbf{E} + ... \tag{1.27}$$

where $\mathbf{E}$ is the electric field and χ_1 is the linear susceptibility. The parameters χ_2 and χ_3 are second and third order nonlinear susceptibilities respectively. Presence of the dielectric medium contributes to the electromagnetic energy density. This contribution is proportional to $\mathbf{P}.\mathbf{E}$. Therefore, the interaction Hamiltonian H_{emi} is

$$H_{emi} = \lambda_c \mathbf{P}.\mathbf{E} \tag{1.28}$$

where λ_c is the proportionality constant. Now, if the symmetry of the medium is chosen in such a way that the only nonlinear interaction in the medium appears due to the presence of the $(m-1)$-th order susceptibility (χ_{m-1}) and the macroscopic magnetization (if any) is neglected, then the leading contribution to the interaction part of the electromagnetic energy comes through the coupling of the $(m-1)$-th order nonlinear susceptibility. Therefore, the interaction energy is proportional to the m-th power of the electric field. Now using (1.24-1.28) the total Hamiltonian of the system which is a sum of free field and interaction Hamiltonians, can be written as

$$\begin{array}{rcl} H & = & a^\dagger a + \frac{1}{2} + \frac{\lambda}{m(2)^{\frac{m}{2}}}(a^\dagger + a)^m \\ & = & \frac{X^2}{2} + \frac{\dot{X}^2}{2} + \frac{\lambda}{m}X^m \, . \end{array} \tag{1.29}$$

The parameter λ is the coupling constant and is a function of the $(m-1)$-th order nonlinear susceptibility (χ_{m-1}) of the medium. The above Hamiltonian (1.29) represents a generalized anharmonic oscillator of unit mass and unit frequency. The equation of motion corresponding to (1.29) is

$$\ddot{X} + X + \lambda X^{m-1} = 0 \tag{1.30}$$

which can not be solved exactly for $m > 2$. Present work provides a first order operator solution for this generalized anharmonic oscillator and the solution is used here to study various nonclassical properties of radiation field interacting with a nonlinear medium.

1.1.2.1 Special cases of the model Hamiltonian

The Hamiltonian (1.29) represents a class of physical systems. Among those systems $m = 4$ is a case of particular interest because even order susceptibilities (χ_2, χ_4 etc.) would vanish for an inversion symmetric medium. Hence the leading contribution to the nonlinear polarization in an inversion symmetric medium comes through the third order susceptibility (χ_3) and the corresponding Hamiltonian is

$$\begin{array}{rcl} H & = & \frac{X^2}{2} + \frac{\dot{X}^2}{2} + \frac{\lambda}{4}X^4 \\ & = & a^\dagger a + \frac{\lambda}{16}(a^\dagger + a)^4 \, . \end{array} \tag{1.31}$$

This is the Hamiltonian of a quartic anharmonic oscillator whose equation of motion is

$$\ddot{X} + X + \lambda X^3 = 0. \tag{1.32}$$

Equation (1.32) is very important in the context of quantum optics, nonlinear optics, molecular physics, ϕ^4 field theory and many other branches of physics. But equation (1.32) is not exactly solvable due to the presence of the off-diagonal terms in the Hamiltonian (1.31). These off-diagonal interaction terms are called photon number non conserving terms[1]. The interaction involving the operator a^2 ($a^{\dagger 2}$) may be viewed as the loss (gain) of two photons. On the

[1]According to Loudon [3] the off-diagonal terms are called energy nonconserving but Gerry [4] mentioned them as photon number nonconserving terms. The phrase 'photon number nonconserving' appears to be more appropriate in the context of the physical process involved. So we mention the offdiagonal terms as photon number nonconserving terms.

other hand, interaction involving $a^\dagger a$ keeps the photon number conserved (i.e one photon is created and one photon is destroyed). So the terms proportional to a^2, $a^{\dagger 2}$, $a^\dagger a^3$, $a^{\dagger 3}a$, a^4 and $a^{\dagger 4}$ are photon number nonconserving in nature. It is usually assumed that the photon number nonconserving terms have no significant contribution in the time development of the field operators. This assumption may be realized in the following way. In the absence of interaction, the time development of the annihilation operator $a(t) = a(0)e^{-it}$. Thus the terms proportional to a^2, $a^{\dagger 2}$, $a^\dagger a^3$, $a^{\dagger 3}a$, a^4 and $a^{\dagger 4}$ are rapidly oscillating compared to the terms $a^\dagger a$ and $a^{\dagger 2}a^2$. These rapidly oscillating terms contribute little to the interaction Hamiltonian. The assumption is widely used and is called the Rotating Wave Approximation (RWA). Under RWA (1.31) assumes an extremely simple and exactly solvable form

$$H = a^\dagger a + \lambda_g a^{\dagger^2} a^2. \tag{1.33}$$

Gerry [5] and Lynch [6] used this rotating wave approximated Hamiltonian (1.33) to study the quantum phase fluctuations of coherent light interacting with a third order nonlinear medium. Joshi *et al* [7] used the Hamiltonian (1.33) to study the sensitivity of nonadiabatic geometric phase on the initial photon statistics in a dispersive fiber. Tanas [8] used the same Hamiltonian (1.33) to study the possibilities of generating squeezed state in the interaction of single mode electromagnetic field with a nonabsorbing nonlinear medium.

The RWA is hardly valid if the noncatalytic nonlinearities are present in the system [9]. For example, the term proportional to $a^{\dagger 4}$ is responsible for higher harmonic generation and occurs naturally corresponding to the field interacting with a nonlinear medium. In fact, the terms proportional to a^4 and $a^{\dagger 4}$ were taken into consideration by Tombesi and Mecozzi [10] to study the possibilities of squeezing (of coherent light passing through a nonlinear medium). Again the nonconserving energy terms are responsible for the well known Bloch-Siegert shift if the field frequency is low (e.g RF and MW) [11].

Keeping these facts in mind, in the present work, we include the offdiagonal terms in the model Hamiltonian and observe that important physical information is lost under RWA calculations. In spite of working with χ_3 medium only, here we study $(m-1)$-th order nonlinear medium in general with particular emphasis to the third order nonlinear medium, i.e. to the case of a quartic anharmonic oscillator.

Depending upon the order of the anharmonicity the anharmonic oscillators are identified by different names such as quartic and sextic anharmonic oscillators. Hamiltonian of these oscillators, their equations of motion and classical counter parts of the equations of motion are required by us at different stages of the development of the present work. So here we give a brief table, containing this information.

Anharmonic oscillator	Classical Hamiltonian	Classical equation of motion	Quantum Hamiltonian	Quantum equation of motion	Nonvanishing susceptibility
Generalized	$\frac{x^2}{2}+\frac{\dot{x}^2}{2}+\frac{\lambda}{m}x^m$	$\ddot{x}+x+\lambda x^{m-1}=0$	$\frac{X^2}{2}+\frac{\dot{X}^2}{2}+\frac{\lambda}{m}X^m$	$\ddot{X}+X+\lambda X^{m-1}=0$	χ_{m-1}
Quartic	$\frac{x^2}{2}+\frac{\dot{x}^2}{2}+\frac{\lambda}{4}x^4$	$\ddot{x}+x+\lambda x^3=0$	$\frac{X^2}{2}+\frac{\dot{X}^2}{2}+\frac{\lambda}{4}X^4$	$\ddot{X}+X+\lambda X^3=0$	χ_3
Sextic	$\frac{x^2}{2}+\frac{\dot{x}^2}{2}+\frac{\lambda}{6}x^6$	$\ddot{x}+x+\lambda x^5=0$	$\frac{X^2}{2}+\frac{\dot{X}^2}{2}+\frac{\lambda}{6}X^6$	$\ddot{X}+X+\lambda X^5=0$	χ_5
Octic	$\frac{x^2}{2}+\frac{\dot{x}^2}{2}+\frac{\lambda}{8}x^8$	$\ddot{x}+x+\lambda x^7=0$	$\frac{X^2}{2}+\frac{\dot{X}^2}{2}+\frac{\lambda}{8}X^8$	$\ddot{X}+X+\lambda X^7=0$	χ_7

Table 1.1

The model Hamiltonian (1.29) is not exactly solvable in a closed analytical form. So we have two alternatives, either we have to approximate the Hamiltonian itself by neglecting some terms (as it is done in RWA) or we have to find approximate solution of the equation of motion (1.30) corresponding to the Hamiltonian (1.29). The first option is already tested for χ_3 medium by various authors [5-8]. Therefore, here we approach the problem by using the second option. The approximate solution can be obtained by different approaches. In the following we discuss the different possible approaches to the AHO model.

1.2 Different approaches to the model

In the last section interaction of an intense electromagnetic field with an $(m-1)$-th order nonlinear medium is modeled by a generalized anharmonic oscillator. The equation of motion corresponding to this model Hamiltonian is not exactly solvable and, in-fact, this provides one of the simplest examples of quantum mechanical systems which can not be solved without making use of approximation techniques. Approximate solution of the AHO problem can be obtained in three different pictures in quantum mechanics. These approaches are called Schrödinger picture, Heisenberg picture and interaction picture. At first we discuss these approaches briefly and after that we choose the appropriate one for the further development of the present study.

1.2.1 Schrödinger picture

The Schrödinger picture deals with the time development of the wave function. Basically, the Schrödinger picture is used to solve the quantum oscillators as eigenvalue problems. In these problems, energy eigenvalues are expressed as the sum of different orders of anharmonic constant λ. However, the energy eigenvalues are found to diverge for large anharmonic constant. In case of small λ, the eigenvalues for different orders can be summed up and the convergence of these sums are ensured by the Borel summability [12, 13] and/or Stieljet conditions [14]. Schrödinger approach is found very successful to treat the quantum anharmonic oscillator problem and there exists an extensive amount of literature on this approach [12-26]. Schrödinger approach is also successfully used to enrich the subject of large order perturbation theory [15-19], divergent expansion of quantum mechanics [12, 20 and 27], Laplace transformation representation of energy eigenvalues [21] and computational physics [22-25].

1.2.2 Heisenberg picture

In this approach, one solves Heisenberg equation of motion to obtain the time evolution of the operators. But the noncommuting nature of the operators poses serious difficulties to obtain approximate operator solutions of the quantum anharmonic oscillators. Only few methods are available till date. In 1967 Aks [28] obtained an operator solution of a quantum quartic anharmonic oscillator (QQAHO) by using the renormalization technique. In 1970 Aks and Caharat [29] reported an extension of the earlier work by using the method of Bogoliubov and Krylov. But the recent interest in the problem started with the work of Bender and Bettencourt [30-31] who obtained the solution of a QQAHO using multiscale perturbation theory (MSPT) in 1996. Later on Mandal [32] proposed a Taylor series approach, Egusquiza and Basagoiti [33] developed renormalization technique, Kahn and Zarmi [34] used a near identity transform method to solve the QQAHO problem. Fernández [35,36] used an eigenvalue approach to solve QQAHO.

Some of the recently derived solutions are obtained as a part of the present work [37-40] and are used here to study quantum optical and nonlinear optical effects. Therefore, these solutions are much important in the context of the present thesis. Keeping that in mind, here we give a list of the solutions of AHO problem available in Heisenberg picture.

Authors	Year	Method	Oscillator
Aks [28]	1967	Renormalization	Quartic
Aks and Caharat [29]	1969	Bogoliubov and Krylov	Quartic
Bender and Bettencourt [30]	1996	MSPT	Quartic
Mandal [32]	1998	Taylor series	Quartic
Egusquiza and Basagoiti [33]	1998	Renormalization Group	Quartic
Kahn and Zarmi [34]	1999	Near identity transform	Quartic
Speliotopoulos [41]	2000	nonperturbative	Quartic
Pathak [37]	2000	MSPT	Generalized
Pathak and Mandal [38]	2001	Taylor series	Quartic
Pathak and Mandal [39]	2001	Taylor Series	Sextic and Octic
Fernández [35]	2001	eigenvalue approach	Quartic
Fernández [36]	2001	eigenvalue approach	Particular values of m
Pathak and Fernández [40]	2001	All the existing methods	Generalized

Table 1.2

1.2.3 Interaction picture

In this approach the time dependence of the physically measurable quantity is described by letting both observable and state vary with time. The observables are rotated in one direction with a transformation generated by part of the Hamiltonian, and the states are rotated in the opposite direction by a transformation generated by the other part (the interaction part) of the Hamiltonian. In case of anharmonic oscillator there is a simple relation between the annihilation operator obtained in Heisenberg approach and that obtained in interaction picture. Only one solution [40] of general anharmonic oscillator in this picture is reported till date.

1.3 Quantum fluctuations and appropriate approach for our study

Second order variance $(\triangle A)^2 =< A^2 > - < A >^2$ is a measure of quantum fluctuations associated with an arbitrary quantum mechanical observable A. In the present work quantum fluctuations (variance) are calculated with respect to initial the coherent state $|\alpha >$. There are substantial quantum fluctuations even in a pure coherent state. For example, the quadrature fluctuation in $|\alpha >$ is

$$(\triangle X)^2 = \frac{1}{2}\, and\, (\triangle \dot{X})^2 = \frac{1}{2} \quad (1.34)$$

while the fluctuation in photon number is

$$(\triangle N)^2 =< N >= |\alpha|^2 \quad (1.35)$$

where $N = a^\dagger a$, is the number operator. Further randomization in the case of general classical states which are random superposition of coherent states can only increase theses fluctuations. But there are some phenomena in which quantum fluctuations reduce below the coherent state level. Let us give some examples,

Example 1: An electromagnetic field is said to be electrically squeezed field if uncertainties in the quadrature phase observable X is less than the coherent state level (i.e. $(\triangle X)^2 < \frac{1}{2}$).

Correspondingly a magnetically squeezed field is one for which $(\triangle\dot{X})^2 < \frac{1}{2}$.
Example 2: In the phenomenon of antibunching (single mode) the fluctuation in photon number is reduced below the Poisson level (1.35),

$$(\triangle N)^2 << N > . \tag{1.36}$$

Example 3: The usual parameters used for the calculation of the quantum phase fluctuations are defined as [5-6]

$$U\left(\theta, t, |\alpha|^2\right) = (\Delta N)^2 \left[(\Delta S)^2 + (\Delta C)^2\right] \Big/ \left[< S >^2 + < C >^2\right] \tag{1.37}$$

$$S\left(\theta, t, |\alpha|^2\right) = (\Delta N)^2 (\Delta S)^2 \tag{1.38}$$

and

$$Q\left(\theta, t, |\alpha|^2\right) = S\left(\theta, t, |\alpha|^2\right) \Big/ < C >^2 \tag{1.39}$$

where S and C are sine and cosine operators respectively (explicit definitions are given in chapter 3). Reduction of these quantum phase fluctuation parameters from their initial values is possible for some particular values of θ and t. When $S(\theta, t, |\alpha|^2)$ gets reduced to below its coherent state value then at least one nonclassical effect (either magnetically squeezed electromagnetic field or antibunching of photons or both) is observed.

Reduced fluctuation states can not be represented by P-representation. Therefore, they are in the paradigm of nonclassical states. In the present thesis we study the possibilities of observing these nonclassical states in the interaction of a coherent electromagnetic field with a nonlinear medium. From the above examples it is clear that in order to study the possibilities of observing nonclassical states we have to calculate quantum fluctuations in different observables (e.g. $N, S, C, X, \dot{X}$ etc.). Again if we have to calculate quantum fluctuations in any of these observables then we have to know the time evolution of creation and annihilation operators. Therefore, we have to obtain operator solution of the equation of motion of the anharmonic oscillator. This is why, we have to work either in Heisenberg picture or interaction picture. In the next chapter we obtain the operator solutions and use them to study different quantum optical and nonlinear optical phenomena in the subsequent chapters.

Chapter 2 deals with the operator solution of the equations of motion (1.30) corresponding to our model Hamiltonian (1.29). The works reported in this chapter are arranged according to the time of their appearance to give a clear exposure of the time development of the subject as well as that of our work. For example, at first we report classical and quantum solutions of quartic, sextic and octic anharmonic oscillators by using Taylor series technique. While working with Taylor series, we observe that the approach is lengthy and time consuming. But many intrinsic symmetries of the problem is made transparent to us. For example, we observe that first order correction to the frequency operator is always a function of the unperturbed Hamiltonian (H_0)[2]. Using these observations and imposing a physical condition that a correct solution should give correct frequency shift we have succeeded to generalize the results of Bender and Bettencourt [30] and have obtained the solution of the generalized AHO [37]. Later on we have generalized the quartic oscillator solutions obtained by various other techniques and observed that an apparent discrepancy is present between the solutions obtained by different techniques. Then the question arises: Which solution should be used for our physical calculations? Therefore, at the

[2] An extensive proof of this observation is given by Speliotopoulos [41].

end of chapter 2 we compare different solutions and conclude that all the correct solutions are equivalent and the apparent discrepancy is due to the use of different ordering of the operators. Showing this equivalence we become mathematically equipped to investigate the possibilities of observing different nonlinear optical phenomena in a nonlinear dielectric medium.

In chapter 3 we study quantum phase fluctuations of coherent light. We begin by asking the question: How can one write down a quantum mechanical operator corresponding to the phase of a harmonic oscillator or equivalently a single mode of electromagnetic field? This question is the statement of the quantum phase problem. Since Dirac had introduced this problem in 1927 [42] many people tried to provide a satisfactory Hermitian phase operator. As a result of these attempts there exist different definitions of the phase operator. In chapter 3 we shortly review a few of them and choose Pegg Barnett approach [43, 44] to calculate the quantum phase fluctuation parameters $S(\theta, t, |\alpha|^2), Q(\theta, t, |\alpha|^2)$ and $U(\theta, t, |\alpha|^2)$ by using the operator solutions of quartic anharmonic oscillator derived in chapter2. These fluctuation parameters were already studied by Gerry [5] and Lynch [6] for RW approximated Hamiltonian (1.33) of the same system. Here we observe that enhancement as well as reduction of $S(\theta, t, |\alpha|^2)$, $Q(\theta, t, |\alpha|^2)$ and $U(\theta, t, |\alpha|^2)$ with $|\alpha|^2$ (number of photons present before the interaction) is possible for different values of the free evolution time t and phase θ of the input coherent light. This observation is in sharp contrast with the earlier results [5, 6]. Thus the importance of inclusion of photon number nonconserving terms in the model Hamiltonian is established. This interesting observation of chapter 3 has inspired us to study the other nonlinear optical effects with the same model.

In chapter 4 we study the possibilities of generating electromagnetically squeezed light in a third order nonlinear medium. At first an analytic expression for the quantum fluctuation in phase quadrature X is obtained (up to the second power in λ). We observe that an electrically squeezed electromagnetic field can be produced due to the interaction of an intense beam of coherent light with a third order nonlinear medium. A few special cases are discussed.

In chapter 5 quantum statistical properties of the radiation field is discussed. To be more precise, bunching and antibunching of photons and the photon number distribution (PND) is studied. We observe that we can obtain nonclassical phenomenon of antibunching due to the interaction of an electromagnetic field with third order nonlinear medium of inversion symmetry for particular values of interaction time and phase of the input coherent state.

In chapter 6 our studies on third order nonlinear medium is extended to the $(m-1)$-th order nonlinear medium in general. We study the nonadiabatic geometric phase or Aharonov Anandan phase for our physical system. Our works of chapter 4 and 5 are extended to the case of $(m-1)$-th order nonlinear medium. Possibilities of generating nonclassical states in an $(m-1)$-th order nonlinear medium are also discussed.

Finally in chapter 7 an outlook on the present work is given. In particular we talk about the limitations of our endeavor and future scope of investigations along the line of thought indicated by us.

Chapter 2

Mathematical preliminaries

Physical scientists have always tried to understand the Nature in terms of simple models. The simple harmonic oscillator (SHO) is perhaps the most useful one among them. A particle subject to a restoring force proportional to its displacement gives rise to the model of a SHO. The Hamiltonian corresponding to a classical SHO of unit mass and unit frequency is given by

$$H = \frac{p^2}{2} + \frac{x^2}{2}\,. \tag{2.1}$$

The Hamilton's equations for the SHO are

$$\begin{aligned} \dot{p} &= -\tfrac{\partial H}{\partial x} = -x \\ \dot{x} &= \tfrac{\partial H}{\partial p} = p\,. \end{aligned} \tag{2.2}$$

Therefore, the equation of motion of the SHO is

$$\ddot{x} + x = 0 \tag{2.3}$$

where equation (2.2) is used. The solution of the equation (2.3) is found to be

$$x_0(t) = x(0)\cos t + \dot{x}(0)\sin t. \tag{2.4}$$

The parameters $x(0)$ and $\dot{x}(0)$ are the initial position and momentum of the oscillator. The subscript '0' denotes the zeroth order (i.e $\lambda = 0$) solution. Thus the position of the oscillator at a later time t is completely known in terms of the initial position and momentum. Nevertheless, the momentum is also known from the Hamilton's equations (2.2). Hence, the oscillator problem is completely solved.

The equation of motion of a quantum SHO may simply be obtained by imposing x and $\dot{x}$ as noncommuting operators and the solution of a quantum SHO in Heisenberg picture is the operator equivalent of the equation (2.4).

Now for the real physical problems, anharmonicity and/or damping are to be incorporated in the model Hamiltonian and hence in the equation of motion. However, in the previous chapter we have seen that only anharmonic term appears in the model Hamiltonian of our physical system. The general form of the Hamiltonian of a classical anharmonic oscillator having unit mass and unit frequency is given by

$$H = \frac{p^2}{2} + \frac{x^2}{2} + \frac{\lambda}{m}x^m \tag{2.5}$$

where $m\,(\geq 3)$ is an integer and λ is the anharmonic constant. Depending upon the problem of physical interest, different types of anharmonic oscillators appear. Of course, the anharmonic constant λ is different for different types of anharmonic oscillators. The equation of motion corresponding to the Hamiltonian (2.5) is

$$\ddot{x} + x + \lambda x^{m-1} = 0. \tag{2.6}$$

where the equation (2.2) is used. We need an operator solution of (1.30) to study the quantum fluctuations of coherent light in nonlinear media. But equation of motion (1.30) and its classical counterpart (2.6) are not exactly solvable in a closed analytical form[1]. Keeping this in view, present chapter is devoted to the study of classical and quantum anharmonic oscillator problem. This problem is already investigated by several authors. To begin with, we recall the problem of a classical quartic oscillator (Duffing oscillator).

2.1 Classical and quantum solutions of the quartic oscillator

For $m = 4$, equation (2.6) has an exact solution in terms of the elliptic function [45]. This solution is available only in the phase plane. For positive anharmonic constant $(\lambda > 0)$, the solution is periodic with a fixed center. The period is also obtained in terms of the elliptic functions. However, this approach is not useful in many occasions. For example, it is not possible to predict the trajectory of the particle executing quartic anharmonic motion. For this reason, the solution in the phase plane is hardly useful except in the nonlinear dynamical studies. Keeping the above facts in view, a large number of approximate methods have been devised for the purpose of getting an analytical solution of the classical Duffing oscillator problem. These include perturbation technique [45], variation of parameters [46] and Taylor series approaches [47]. The ordinary perturbation technique, a pedestrian approach, leads to the unwanted secular terms [45]. The removal of secular terms from the solution is a serious problem. There are some methods which are successfully used to sum up the secular terms for all orders. These include “tucking in technique” [48], multiscale perturbation theory (MSPT) and renormalization technique [45].

The noncommuting nature of the operators poses serious difficulties for the purpose of getting approximate operator solutions to the quantum anharmonic oscillators. However, few methods are available for the purpose. We have already mentioned these methods [28-41] in 1.2.2. In the next two subsections of the present thesis we derive solutions of classical and quantum quartic anharmonic oscillators respectively. Actually, at first we develop a second order solution for the QQAHO using the Taylor series approach and after that, we construct a first order solution for $m - th$ oscillator in general using various other techniques. Solutions obtained by different techniques are compared in the subsequent subsections. The solutions are used to study the interaction of the strong electromagnetic field with a dielectric medium in the later part of the thesis.

2.1.1 Classical and quantum solution of quartic oscillator

Let us attempt a second order solution of the QQAHO by using the Taylor series approach devised by Mandal [32]. At first we develop the solution of the QAHO in classical picture and after that a straight forward generalization is made for getting the corresponding solution in quantum picture.

[1]Exact solution of (2.6) can be obtained in terms of elliptic functions. However, that solution can not be used to predict the trajectory of the particle executing anharmonic motion. So the solution obtained in terms of elliptic functions are not exact in a strict sense.

The differential equation of the classical Duffing oscillator of unit mass and unit frequency is

$$\ddot{x} + x + \lambda x^3 = 0 \tag{2.7}$$

where the parameter λ is the anharmonic constant of the system. The solution of the equation (2.7) may be written as the sum of different orders of anharmonicities. Thus the corresponding solution is

$$x(t) = x_0(t) + x_1(t) + x_2(t) + \ldots \tag{2.8}$$

where $x_0(t)$, $x_1(t)$ and $x_2(t)$ are zeroth, first and second order solutions respectively. The first and second order solutions are proportional to λ and λ^2 respectively. The zeroth and first order solutions are available from the works of Mandal [32] and are given by

$$x_0(t) = x(0)\cos t + \dot{x}(0)\sin t \tag{2.9}$$

and

$$\begin{aligned} x_1(t) &= -\tfrac{\lambda x^3(0)}{32}(\cos t - \cos 3t + 12t\sin t) \\ &+ \tfrac{3\lambda x^2(0)\dot{x}(0)}{32}(\sin 3t - 7\sin t + 4t\cos t) \\ &- \tfrac{3\lambda x(0)\dot{x}^2(0)}{32}(\cos 3t - \cos t + 4t\sin t) \\ &- \tfrac{\dot{x}^3(0)}{32}(\sin 3t + 9\sin t - 12t\cos t) \end{aligned} \tag{2.10}$$

respectively. The purpose of the present subsection is to find the second order solution $x_2(t)$ of the classical quartic oscillator and to introduce the Taylor series method which is also used here to deal with the higher anharmonic oscillators. The solution of the classical Duffing oscillator (without forcing term) is given by the following Taylor series [47]

$$x(t) = x(0) + t\dot{x}(0) + \frac{t^2}{2!}\ddot{x}(0) + \ldots\ . \tag{2.11}$$

The assumed Taylor series is expanded about the origin. Furthermore, we assume that t is sufficiently small and is such that the above series (2.11) expansion is possible. Now we express higher order time derivatives of $x(t)$ at $t = 0$

$$\begin{aligned} \ddot{x}^{\cdot}(0) &= -\dot{x}(0) - 3\lambda x^2(0)\dot{x}(0) \\ \dot{x}^{\cdots}(0) &= x(0) + 4\lambda x^3(0) - 6\lambda x(0)\dot{x}^2(0) + 3\lambda^2 x^5(0) \\ \ddot{x}^{\cdots}(0) &= \dot{x}(0) + 24\lambda x^2(0)\dot{x}(0) - 6\lambda\dot{x}^3(0) + 27\lambda^2 x^4(0)\dot{x}(0) \\ \ldots &\ldots \ldots \\ \ldots &\ldots \ldots \end{aligned} \tag{2.12}$$

where we neglect the terms beyond λ^2 and always substitute $\ddot{x}$ by $-x - \lambda x^3$. The relations (2.12) are substituted back in equation (2.11) and the coefficients of $x^5(0)$, $x^4(0)\dot{x}(0)$, $x^3(0)\dot{x}^2(0)$, $x^2(0)\dot{x}^3(0)$, $x(0)\dot{x}^4(0)$ and $\dot{x}^5(0)$ are collected. The corresponding coefficients are

$$\begin{aligned} C_1 &= \lambda^2(3 - 51 + 846 - 15078 + 300705 - 6633081 + \ldots) \\ C_2 &= \lambda^2(27 - 639 + 13230 - 284094 + \ldots) \\ C_3 &= \lambda^2(126 - 3888 + 98820 - 2440224 + \ldots) \\ C_4 &= \lambda^2(378 - 14688 + 429084 - 11416464 + \ldots) \\ C_5 &= \lambda^2(756 - 33156 + 1023948 - 27952668 + \ldots) \\ &and \\ C_6 &= \lambda^2(756 - 33156 + 1023948 - 27952668 + \ldots) \end{aligned} \tag{2.13}$$

where the contributions from Taylor expansion part are not taken into account. These contributions are taken care of in the later part of this work. Constructing difference equations for the above series and solving them we can obtain the $r-th$ term of the coefficient C_n. For example, the $r-th$ term of C_1 is given by

$$t'_r \;=\; (-1)^{r+1} \times \tfrac{\lambda^2}{1024}\left(25^{r+1} + 216r \times 9^r + 288r^2 + 240r - 25\right). \tag{2.14}$$

Now we can write the $r-th$ term of the coefficient of $x^5(0)$ as

$$t_r = t'_r \times \frac{t^{2r+2}}{(2r+2)!}, \tag{2.15}$$

where the factor $\frac{t^{2r+2}}{(2r+2)!}$ comes from the Taylor expansion part. The net coefficient of $x^5(0)$ is obtained as

$$\begin{aligned} K_1 = \textstyle\sum_{r=0}^{\infty} t_r \;&=\; \tfrac{\lambda^2}{1024}(\cos 5t - 36\, t \sin 3t - 24\, \cos 3t \\ &- \; 72\, t^2 \cos t + 96\, t \sin t + 23\, \cos t). \end{aligned} \tag{2.16}$$

The parameter K_1 exhibits fifth and third harmonic generations. The remaining coefficients are also obtained by using the similar procedure as adopted for the evaluation of K_1. The corresponding coefficients are

$$\begin{aligned} K_2 &= \tfrac{\lambda^2}{1024}(5 \sin 5t + 108t \cos 3t - 132 \sin 3t - 72t^2 \sin t + 599 \sin t - 336t \cos t) \\ K_3 &= \tfrac{2\lambda^2}{1024}(-5 \cos 5t + 90 \cos 3t + 36t \sin 3t - 72t^2 \cos t - 85 \cos t + 264t \sin t) \\ K_4 &= \tfrac{2\lambda^2}{1024}(-5 \sin 5t + 36t \cos 3t + 6 \sin 3t - 72t^2 \sin t + 427 \sin t - 456t \cos t) \\ K_5 &= \tfrac{\lambda^2}{1024}(5 \cos 5t + 108t \sin 3t + 108 \cos 3t - 72t^2 \cos t - 113 \cos t + 240t \sin t) \\ &and \\ K_6 &= \tfrac{\lambda^2}{1024}(\sin 5t - 36t \cos 3t + 48 \sin 3t - 72t^2 \sin t + 271 \sin t - 384t \cos t)\,. \end{aligned} \tag{2.17}$$

Hence the desired second order solution is given by

$$\begin{aligned} x_2(t) \;&=\; K_1 x^5(0) + K_2 x^4(0)\dot{x}(0) + K_3 x^3(0)\dot{x}^2(0) \\ &+ \; K_4 x^2(0)\dot{x}^3(0) + K_5 x(0)\dot{x}^4(0) + K_6 \dot{x}^5(0). \end{aligned} \tag{2.18}$$

The total solution (2.8) is simply the sum of (2.9), (2.10) and (2.18). For $\dot{x}(0) = 0$ and $x(0) = A$, the total solution reduces to an extraordinarily simple form

$$\begin{aligned} x(t) \;&=\; A \cos t - \tfrac{\lambda A^3}{32}(\cos t - \cos 3t + 12t \sin t) + \tfrac{\lambda^2 A^5}{1024}(\cos 5t \\ &- \; 36\, t \sin 3t - 24\, \cos 3t - 72\, t^2 \cos t + 96\, t \sin t + 23\, \cos t). \end{aligned} \tag{2.19}$$

It is clear that the presence of the secular terms proportional to $t \sin t$, $t \sin 3t$ and $t^2 \cos t$ pose a serious difficulty as t grows. For weak coupling case $(i.e\, \lambda \ll 1)$, these terms may be summed up with the help of various summation techniques. However, for strong coupling the summation technique would fail and would give divergent solutions. We confine ourselves in the weak coupling regime and hence the summation for different orders of λ is possible. For weak coupling, the secular term in the first order solution is known to produce the frequency shift of the oscillator [32]. In the present case similar frequency shift may be observed in following way. Equation (2.19) can be rearranged as

$$\begin{aligned} x(t) \;&=\; A\left[1 - \tfrac{(\frac{3}{8}\lambda A^2 t)^2}{2!}\right]\cos t - A(\tfrac{3}{8}\lambda A^2 t)\sin t - \tfrac{\lambda A^3}{32}\Big[(\cos t - \tfrac{3}{8}\lambda A^2 t \sin t) \\ &- \; (\cos 3t - \tfrac{9}{8}\lambda A^2 t \sin 3t)\Big] + \tfrac{\lambda^2 A^5}{1024}[\cos 5t - 24\, \cos 3t + 23\, \cos t] + \tfrac{21\lambda^2 A^5}{256} t \sin t. \end{aligned} \tag{2.20}$$

Now we use the tucking in technique to remove the secular terms. As we are dealing with the second order solution we can write

$$\begin{array}{rcl} \sin(b\lambda t) & = & b\lambda t \\ \cos(b\lambda t) & = & 1-\frac{(b\lambda t)^2}{2!} \\ (b'\lambda)\sin(b\lambda t) & = & b'b\lambda^2 t \\ (b'\lambda)\cos(b\lambda t) & = & b'\lambda \end{array} \tag{2.21}$$

where b and b' are constants and the terms beyond λ^2 are neglected. Thus the secular terms can be removed from the solution and finally the solution (2.20) reduces to

$$x(t) = A\cos\omega^/ t - \frac{\lambda A^3}{32}(\cos\omega^/ t - \cos 3\omega^/ t) + \frac{\lambda^2 A^5}{1024}(\cos 5\omega^/ t - 24\cos 3\omega' t + 23\cos\omega^/ t). \tag{2.22}$$

The shifted frequency is given by

$$\omega^/ = 1 + \frac{3}{8}\lambda A^2 - \frac{21}{256}\lambda^2 A^4 \tag{2.23}$$

which coincides exactly with the frequency obtained by using Laplace transform [49]. Thus, the existing second order solution of a QAHO [49] is reproduced as a special case $(i.e\, \dot{x}(0) = 0,\, x(0) = A)$ of our solution (2.8). For $\dot{x}(0) = 0, x(0) = A$ the frequency shift of the oscillator due to quartic anharmonicity is $(\frac{A^2}{8} - \frac{7\lambda A^4}{256})3\lambda$. The frequency shift due to the first term $(3\lambda A^2/8)$ dominates over the frequency shift due to the second term $(21\lambda^2 A^4/256)$ as long as λ is small. Hence, the shift in the frequency increases with the increase of λ till the contribution of the second term compensates the contribution of the first one. Further increase of λ causes a frequency shift in the opposite direction. Thus the second order correction to the frequency shift is quite consistent with the physical system. Now to have an idea of how good the solutions (2.19) and (2.22) are we compare them with the exact numerical solution obtained by Mathematica (figure 1). From figure 1, we observe that the first order frequency renormalized solution (2.22) coincides exactly with the exact numerical solution obtained by using Mathematica (for $a = 2$ and $\lambda = 0.05$). However, the solution (2.19) diverges with the increase of time (figure 1). The divergent nature of the solution is a manifestation of the presence of secular term.

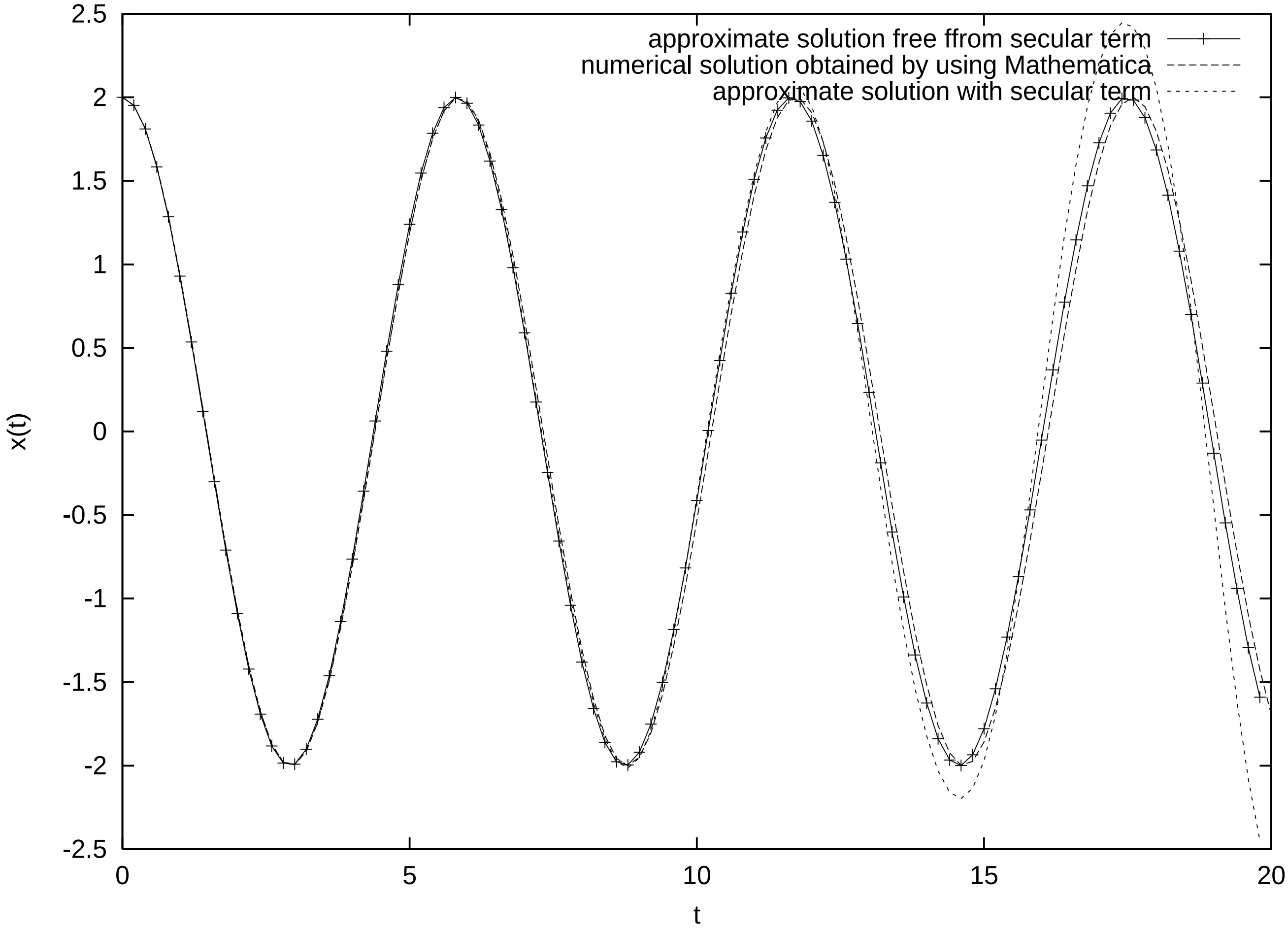

figure 1

2.1.2 Solution of quantum quartic oscillator

It is straightforward to extend the classical solution (2.8) into a quantum solution for the anharmonic oscillator by using the condition of quantization [50]. The corresponding solution is given by

$$
\begin{aligned}
X(t) \;=\; & X(0)\,\cos t + \dot X(0)\,\sin t - \tfrac{\lambda X^3(0)}{32}(\cos t - \cos 3t + 12t\sin t)\\
+\; & \tfrac{\lambda}{32}[X^2(0)\dot X(0) + X(0)\dot X(0)X(0) + \dot X(0)X^2(0)] \times (\sin 3t - 7\sin t + 4t\cos t)\\
-\; & \tfrac{\lambda}{32}[X(0)\dot X^2(0) + \dot X(0)X(0)\dot X(0) + \dot X^2(0)X(0)] \times (\cos 3t - \cos t + 4t\sin t)\\
-\; & \tfrac{\lambda \dot X^3(0)}{32}(\sin 3t + 9\sin t - 12t\cos t)\\
+\; & \tfrac{\lambda^2 X^5(0)}{1024}(\cos 5t - 36t\sin 3t - 24\cos 3t - 72t^2\cos t + 96t\sin t + 23\cos t)\\
+\; & \tfrac{\lambda^2}{5120}[X^4(0)\dot X(0) + X^3(0)\dot X(0)X(0) + X^2(0)\dot X(0)X^2(0)\\
+\; & X(0)\dot X(0)X^3(0) + \dot X(0)X^4(0)]\\
\times\; & (5\sin 5t + 108t\cos 3t - 132\sin 3t - 72t^2\sin t + 599\sin t - 336t\cos t)\\
+\; & \tfrac{\lambda^2}{5120}[X^3(0)\dot X^2(0) + X^2(0)\dot X^2(0)X(0) + X^2(0)\dot X(0)X(0)\dot X(0) + X(0)\dot X^2(0)X^2(0)\\
+\; & X(0)\dot X(0)X(0)\dot X(0)X(0) + X(0)\dot X(0)X^2(0)\dot X(0) + \dot X^2(0)X^3(0)\\
+\; & \dot X(0)X(0)\dot X(0)X^2(0) + \dot X(0)X^2(0)\dot X(0)X(0) + \dot X(0)X^3(0)\dot X(0)]\\
\times\; & (-5\cos 5t + 90\cos 3t + 36t\sin 3t - 72t^2\cos t - 85\cos t + 264t\sin t)\\
+\; & \tfrac{\lambda^2}{5120}[\dot X^3(0)X^2(0) + \dot X^2(0)X^2(0)\dot X(0) + \dot X^2(0)X(0)\dot X(0)X(0) + \dot X(0)X^2(0)\dot X^2(0)\\
+\; & \dot X(0)X(0)\dot X(0)X(0)\dot X(0) + \dot X(0)X(0)\dot X^2(0)X(0) + X^2(0)\dot X^3(0)\\
+\; & X(0)\dot X(0)X(0)\dot X^2(0) + X(0)\dot X^2(0)X(0)\dot X(0) + X(0)\dot X^3(0)X(0)]\\
\times\; & (-5\sin 5t + 36t\cos 3t + 6\sin 3t - 72t^2\sin t + 427\sin t - 456t\cos t)\\
+\; & \tfrac{\lambda^2}{5120}[X(0)\dot X^4(0) + \dot X(0)X(0)\dot X^3(0) + \dot X^2(0)X(0)\dot X^2(0) + \dot X^3(0)X(0)\dot X(0) + \dot X^4(0)X(0)]\\
\times\; & (5\cos 5t + 108t\sin 3t + 108\cos 3t - 72t^2\cos t - 113\cos t + 240t\sin t)\\
+\; & \tfrac{\lambda^2 \dot X^5(0)}{1024}(\sin 5t - 36t\cos 3t + 48\sin 3t - 72t^2\sin t + 271\sin t - 384t\cos t)
\end{aligned}
\tag{2.24}
$$

where $X(0)$ and $\dot X(0)$ are the initial position and momentum operators. Of course, the direct use of the Taylor series for the equivalent operator differential equation (1.30) leads to the same solution (2.24). The quantization condition, $[X(t), \dot X(t)] = i$, takes care the passage of classical solution (2.8) to quantum solution (2.24) which may be written in the symmetrical form as

$$
\begin{aligned}
X(t) \;=\; & X(0)\,\cos t + \dot X(0)\,\sin t - \tfrac{\lambda X^3(0)}{32}(\cos t - \cos 3t + 12t\sin t)\\
+\; & \tfrac{3\lambda}{64}[X^2(0)\dot X(0) + \dot X(0)X^2(0)] \times (\sin 3t - 7\sin t + 4t\cos t)\\
-\; & \tfrac{3\lambda}{64}[X(0)\dot X^2(0) + \dot X^2(0)X(0)] \times (\cos 3t - \cos t + 4t\sin t)\\
-\; & \tfrac{\lambda \dot X^3(0)}{32}(\sin 3t + 9\sin t - 12t\cos t)\\
+\; & \tfrac{\lambda^2 X^5(0)}{1024}(\cos 5t - 36t\sin 3t - 24\cos 3t - 72t^2\cos t + 96t\sin t + 23\cos t)\\
+\; & \tfrac{\lambda^2}{2048}[X^4(0)\dot X(0) + \dot X(0)X^4(0)]\\
\times\; & (5\sin 5t + 108t\cos 3t - 132\sin 3t - 72t^2\sin t + 599\sin t - 336t\cos t)\\
+\; & \tfrac{2\lambda^2}{2048}[X^3(0)\dot X^2(0) + \dot X^2(0)X^3(0) + 3X(0)]\\
\times\; & (-5\cos 5t + 90\cos 3t + 36t\sin 3t - 72t^2\cos t - 85\cos t + 264t\sin t)\\
+\; & \tfrac{2\lambda^2}{2048}[\dot X^3(0)X^2(0) + X^2(0)\dot X^3(0) + 3\dot X(0)]\\
\times\; & (-5\sin 5t + 36t\cos 3t + 6\sin 3t - 72t^2\sin t + 427\sin t - 456t\cos t)\\
+\; & \tfrac{\lambda^2}{2048}[X(0)\dot X^4(0) + \dot X^4(0)X(0)]\\
\times\; & (5\cos 5t + 108t\sin 3t + 108\cos 3t - 72t^2\cos t - 113\cos t + 240t\sin t)\\
+\; & \tfrac{\lambda^2 \dot X^5(0)}{1024}(\sin 5t - 36t\cos 3t + 48\sin 3t - 72t^2\sin t + 271\sin t - 384t\cos t).
\end{aligned}
\tag{2.25}
$$

The equation (2.25) is our desired solution for a quantum quartic anharmonic oscillator. The solution contains terms proportional to X^3 and X^5 etcetera. In the classical limit these terms produce the cubic and fifth powers of amplitude respectively. Hence, as a limiting situation, the solution (2.25) gives rise to the solution corresponding to the classical QAHO. It is not at all surprising since the solution (2.25) is obtained from its classical counterpart (2.8) by the imposition of the quantization condition.

The unpleasant secular terms can be removed (summed) from the solution (2.25) by using the tucking in technique. All the secular terms of equation (2.25) are tucked in to have the following solution

$$\begin{array}{rcl}
X(t) & = & \frac{1}{2\cos(\frac{3\lambda t}{8}-\frac{51\lambda^2 nt}{64})}\Big\{X(0)\cos\Psi t+\cos\Psi t\,X(0)+\dot{X}(0)\sin\Psi t+\sin\Psi t\,\dot{X}(0) \\
& - & \frac{\lambda X^3(0)}{32}(\cos\Psi t-\cos 3\Psi t)-\frac{\lambda}{32}(\cos\Psi t-\cos 3\Psi t)X^3(0) \\
& + & \frac{3\lambda}{64}[X^2(0)\dot{X}(0)+\dot{X}(0)X^2(0)]\times(\sin 3\Psi t-7\sin\Psi t)+\frac{3\lambda}{64}(\sin 3\Psi t-7\sin\Psi t) \\
& \times & [X^2(0)\dot{X}(0)+\dot{X}(0)X^2(0)]-\frac{3\lambda}{64}[X(0)\dot{X}^2(0)+\dot{X}^2(0)X(0)]\times(\cos 3\Psi t-\cos\Psi t) \\
& - & \frac{3\lambda}{64}(\cos 3\Psi t-\cos\Psi t)\times[X(0)\dot{X}^2(0)+\dot{X}^2(0)X(0)] \\
& - & \frac{\lambda\dot{X}^3(0)}{32}(\sin 3\Psi t+9\sin\Psi t)-\frac{\lambda}{32}(\sin 3\Psi t+9\sin\Psi t)\dot{X}^3(0) \\
& + & \frac{\lambda^2 X^5(0)}{1024}(\cos 5\Psi t-24\cos 3\Psi t+23\cos\Psi t) \\
& + & \frac{\lambda^2}{1024}(\cos 5\Psi t-24\cos 3\Psi t+23\cos\Psi t)X^5(0) \\
& + & \frac{\lambda^2}{2048}[X^4(0)\dot{X}(0)+\dot{X}(0)X^4(0)]\times(5\sin 5\Psi t-132\sin 3\Psi t+599\sin\Psi t) \\
& + & \frac{\lambda^2}{2048}(5\sin 5\Psi t-132\sin 3\Psi t+599\sin\Psi t)\times[X^4(0)\dot{X}(0)+\dot{X}(0)X^4(0)] \\
& + & \frac{2\lambda^2}{2048}[X^3(0)\dot{X}^2(0)+\dot{X}^2(0)X^3(0)+3X(0)]\times(-5\cos 5\Psi t+90\cos 3\Psi t-85\cos\Psi t) \\
& + & \frac{2\lambda^2}{2048}(-5\cos 5\Psi t+90\cos 3\Psi t-85\cos\Psi t)\times[X^3(0)\dot{X}^2(0)+\dot{X}^2(0)X^3(0)+3X(0)] \\
& + & \frac{2\lambda^2}{2048}[\dot{X}^3(0)X^2(0)+X^2(0)\dot{X}^3(0)]\times(-5\sin 5\Psi t+6\sin 3\Psi t+427\sin\Psi t) \\
& + & \frac{2\lambda^2}{2048}(-5\sin 5\Psi t+6\sin 3\Psi t+427\sin\Psi t)\times[\dot{X}^3(0)X^2(0)+X^2(0)\dot{X}^3(0)] \\
& + & \frac{2\lambda^2}{2048}[3\dot{X}(0)]\times(-5\sin 5\Psi t+6\sin 3\Psi t+403\sin\Psi t) \\
& + & \frac{2\lambda^2}{2048}(-5\sin 5\Psi t+6\sin 3\Psi t+403\sin\Psi t)\times[3\dot{X}(0)] \\
& + & \frac{\lambda^2}{2048}[X(0)\dot{X}^4(0)+\dot{X}^4(0)X(0)]\times(5\cos 5\Psi t+108\cos 3\Psi t-113\cos\Psi t) \\
& + & \frac{\lambda^2}{2048}(5\cos 5\Psi t+108\cos 3\Psi t-113\cos\Psi t)\times[X(0)\dot{X}^4(0)+\dot{X}^4(0)X(0)] \\
& + & \frac{\lambda^2 \dot{X}^5(0)}{1024}(\sin 5\Psi t+48\sin 3\Psi t+271\sin\Psi t) \\
& + & \frac{\lambda^2}{1024}(\sin 5\Psi t+48\sin 3\Psi t+271\sin\Psi t)\dot{X}^5(0)\Big\}
\end{array} \tag{2.26}$$

where n is the eigenvalue of the number operator ($a^\dagger a$). The parameter n may be regarded as the number of photons present in the radiation field corresponding to the electric field operator $X(t)$. The prefactor of the above equation (2.26) takes care of the proper normalization of $X(t)$. The operator Ψ is second order frequency operator and is given by

$$\Psi = 1+\frac{3\lambda H_0}{4}-\frac{\lambda^2}{64}\left(69H_0^2-12X^4(0)+\frac{51}{4}\right) \tag{2.27}$$

where $H_0=\frac{\dot{X}^2(0)}{2}+\frac{X^2(0)}{2}$ is the unperturbed Hamiltonian. Now, the diagonal element of Ψ in the number state basis is

$$\psi_n =< n|\Psi|n >= 1+\frac{3\lambda}{4}(n+\frac{1}{2})-\frac{\lambda^2}{64}(51n^2+51n+21) \tag{2.28}$$

where the following relations are used

$$\begin{array}{rcl}
H_0|n> & = & (n+\frac{1}{2})\,|n> \\
X(0)|n> & = & \frac{1}{\sqrt{2}}\left[\sqrt{n}|n-1>+\sqrt{n+1}|n+1>\right]\ for\, n\neq 0
\end{array}. \tag{2.29}$$

The consequences of our solution (2.26) may be compared critically with the existing results. For example, the energy and hence the frequency shift of a QQAHO is already known from the second order perturbation theory. The energy of a quartic oscillator in $n-th$ state is given by [51]

$$E_n = (n+\frac{1}{2}) + \frac{3\lambda}{8}(n^2+n+\frac{1}{2}) - \frac{\lambda^2}{128}(34n^3+51n^2+59n+21). \tag{2.30}$$

Therefore, the energy difference between two consecutive levels is

$$\Delta E = E_n - E_{n-1} = 1 + \frac{3\lambda n}{4} - \frac{\lambda^2}{64}(51n^2+21). \tag{2.31}$$

In order to calculate the dipole moment matrix element $< n-1|X(t)|n >$ and hence the shifted frequency of the quantum oscillator, we use equations (2.26) and (2.28). The corresponding calculated frequency shift coincides exactly with that of the frequency shift obtained by the perturbation method (2.31). The frequency shift for the vacuum field $(i.e\, n = 0)$ is evidently clear. It actually arises due to the vacuum field interaction. Here we can note that the Lamb shift is also explained in terms of the self interacting field (for a detailed discussion see reference [52]). Actually, our model Hamiltonian permits such an interaction through the quartic anharmonicity. Here we can note that a first order calculation is unable to predict the frequency shift for a vacuum field.

2.1.3 Remarks on the solutions

We obtain an analytical second order solution for a classical quartic anharmonic oscillator by using the Taylor series method. The solution is found to agree with the existing solutions obtained by various methods. The frequency shift of a QAHO increases linearly with the increase of anharmonic constant as long as the first order solution is considered [32, 45]. These type of monotonic increase of the frequency shifts seem to be inconsistent. We obtain a correction to such monotonic increase of frequency shift. The solutions presented here are perturbative. We prefer perturbative solution instead of an exact numerical one. Because they can provide more physical insight into the problem.

The classical solution is quantized in order to obtain the solution corresponding to a quantum quartic oscillator. Interestingly, the frequency shift of the quantum oscillator coincides with that of the frequency shift obtained by using a second order perturbation theory. The second order solution exhibits Lamb shift which is totally absent in a first order solution. Hence, we observe that our solutions reproduce the existing results for classical and quantum oscillators of quartic anharmonicity.

From the mathematical point of view, the solutions presented in this section are of great academic interests. In addition to these, the classical solution finds several applications in nonlinear mechanics. The solution corresponding to the quantum oscillator is useful in quantum optics and in the field theory. For example, the solutions are used to study the quantum statistical properties (e.g. squeezing, higher order squeezing, bunching and antibunching of photons) of radiation field interacting with a cubic nonlinear media in the subsequent chapters of the present thesis. Moreover, the quantum phase of the output field is also studied with the knowledge of field operators $X(t),\ \dot{X}(t),\ a(t)$ and $a^\dagger(t)$.

2.2 Higher anharmonic oscillators : sextic and octic oscillators

In addition to the quartic oscillator, people have studied the higher anharmonic oscillators also [22, 53,54]. These studies are made to obtain the eigenvalues as function of anharmonicity. However, the operator solution of higher anharmonic oscillators were not reported since recent past. Recently we have reported first order operator solutions [39] of sextic and octic anharmonic oscillators by using the Taylor series method. We have also developed a first order solution for the generalized anharmonic oscillator [37]. Fernández obtained first order correction to the frequency operator for a few particular values of m [35]. In the next two subsection of the present thesis, we present analytical solutions of classical and quantum oscillators of sextic and octic anharmonicities by using the Taylor series approach. In the later part of this thesis we discuss the generalized anharmonic oscillator and compare the results with that obtained in the Taylor series approach. The sextic and octic anharmonic oscillators have potential applications in the studies of nonlinear mechanics, molecular physics, quantum optics and in field theory. The solutions are used to obtain the frequency shifts of sextic and octic oscillators. The computed shifts are compared and found to have exact coincidence with the frequency shifts calculated by using a first order perturbation theory.

2.2.1 Classical and quantum solutions of the sextic oscillator

For $m = 6$, the Hamiltonian (2.5) and the equation of motion (2.6) correspond to the case of a sextic anharmonic oscillator. The solution for such an oscillator is obtained as the sum of different orders of anharmonicities. The corresponding solution is

$$x(t) = x_0(t) + x_1(t) + \ldots \tag{2.32}$$

where $x_0(t)$ and $x_1(t)$ are zeroth and first order solutions respectively. The zeroth order solution (2.4) is simply obtained for $\lambda = 0$ in equation (2.6). The purpose of the present section is to find the first order solution $x_1(t)$. By using the Taylor series approach introduced in connection with the quartic oscillator we can obtain first order solution of the classical sextic anharmonic oscillator as

$$x_1(t) = \sum_{i=0}^{5} L_i x^{5-i}(0)\dot{x}^i \tag{2.33}$$

where

$$\begin{aligned}
L_0 &= \tfrac{\lambda}{384}(\cos 5t + 15\cos 3t - 16\cos t - 120\, t\sin t)\\
L_1 &= \tfrac{\lambda}{384}(5\sin 5t + 45\sin 3t - 280\sin t + 120t\cos t)\\
L_2 &= \tfrac{2\lambda}{384}(-5\cos 5t - 15\cos 3t + 20\cos t - 120t\sin t)\\
L_3 &= \tfrac{2\lambda}{384}(-5\sin 5t + 15\sin 3t - 140\sin t + 120t\cos t)\\
L_4 &= \tfrac{\lambda}{384}(5\cos 5t - 45\cos 3t + 40\cos t - 120t\sin t)\\
&and\\
L_5 &= \tfrac{\lambda}{384}(\sin 5t - 15\sin 3t - 80\sin t + 120t\cos t).
\end{aligned} \tag{2.34}$$

The total solution (2.8) is simply the sum of (2.4) and (2.33).

Now to have a check of the solution (2.32), we consider a special case $\dot{x}(0) = 0$ and $x(0) = a$. Hence the total solution reduces to an extraordinarily simple form as

$$x(t) = a\cos t + \frac{\lambda a^5}{384}(\cos 5t + 15\cos 3t - 16\cos t - 120t\sin t). \tag{2.35}$$

It is clear that the presence of a secular term proportional to $t\sin t$ makes life difficult as t grows. We confine ourselves to the weak coupling regime and assume C is a constant such that

$$\begin{array}{rcl} \sin(C\lambda t) &=& C\lambda t \\ \cos(C\lambda t) &=& 1 \end{array} . \tag{2.36}$$

Equation (2.36) is actually a first order form of (2.21). Now, the equation (2.35) can be rearranged as

$$x(t) = a\cos[t(1+\frac{5}{16}\lambda a^4)] + \lambda a^5(-\frac{1}{24}\cos t + \frac{5}{128}\cos 3t + \frac{1}{384}\cos 5t) \tag{2.37}$$

where the equation (2.36) is used. Thus the secular term present in the first order solution (2.35) is summed up for all orders and the frequency shift of the oscillator is obtained. The corresponding frequency shift of the sextic anharmonic oscillator is $\frac{5}{16}\lambda a^4$. The shifted frequency of the oscillator may be viewed as the renormalization of frequency by the anharmonic interaction. Interestingly, the frequency shift obtained in the solution (2.37) exactly coincides with that obtained by Dutt and Lakshmanan [54].

Now, a quantum solution of the sextic anharmonic oscillator may be obtained by replacing $x(t)$ and $\dot{x}(t)$ by noncommuting operators $X(t)$ and $\dot{X}(t)$ respectively and imposing the commutation relation (1.23). Corresponding operator solution of the sextic oscillator is

$$\begin{array}{rcl}
X(t) &=& X(0)\cos t + \dot{X}(0)\sin t + \frac{\lambda}{384}X^5(0)\times(\cos 5t + 15\cos 3t - 16\cos t - 120t\sin t) \\
&+& \frac{\lambda}{1920}\Big[X^4(0)\dot{X}(0) + X^3(0)\dot{X}(0)X(0) \\
&+& X^2(0)\dot{X}(0)X^2(0) + X(0)\dot{X}(0)X^3(0) + \dot{X}(0)X^4(0)\Big] \\
&\times& (5\sin 5t + 45\sin 3t - 280\sin t + 120t\cos t) \\
&+& \frac{\lambda}{1920}\Big[X^3(0)\dot{X}^2(0) + X^2(0)\dot{X}^2(0)X(0) + X^2(0)\dot{X}(0)X(0)\dot{X}(0) \\
&+& X(0)\dot{X}^2(0)X^2(0) + X(0)\dot{X}(0)X(0)\dot{X}(0)X(0) + X(0)\dot{X}(0)X^2(0)\dot{X}(0) \\
&+& \dot{X}^2(0)X^3(0) + \dot{X}(0)X(0)\dot{X}(0)X^2(0) + \dot{X}(0)X^2(0)\dot{X}(0)X(0) + \dot{X}(0)X^3(0)\dot{X}(0)\Big] \\
&\times& (-5\cos 5t - 15\cos 3t + 20\cos t - 120t\sin t) \\
&+& \frac{\lambda}{1920}\Big[\dot{X}^3(0)X^2(0) + \dot{X}^2(0)X^2(0)\dot{X}(0) + \dot{X}^2(0)X(0)\dot{X}(0)X(0) + \dot{X}(0)X^2(0)\dot{X}^2(0) \\
&+& \dot{X}(0)X(0)\dot{X}(0)X(0)\dot{X}(0) + \dot{X}(0)X(0)\dot{X}^2(0)X(0) + X^2(0)\dot{X}^3(0) \\
&+& X(0)\dot{X}(0)X(0)\dot{X}^2(0) + X(0)\dot{X}^2(0)X(0)\dot{X}(0) + X(0)\dot{X}^3(0)X(0)\Big] \\
&\times& (-5\sin 5t + 15\sin 3t - 140\sin t + 120t\cos t) \\
&+& \frac{\lambda}{1920}\Big[X(0)\dot{X}^4(0) + \dot{X}(0)X(0)\dot{X}^3(0) + \dot{X}^2(0)X(0)\dot{X}^2(0) \\
&+& \dot{X}^3(0)X(0)\dot{X}(0) + \dot{X}^4(0)X(0)\Big] \\
&\times& (5\cos 5t - 45\cos 3t + 40\cos t - 120t\sin t) \\
&+& \frac{\lambda\dot{X}^5(0)}{384}(\sin 5t - 15\sin 3t - 80\sin t + 120t\cos t)
\end{array} \tag{2.38}$$

where $X(0)$ and $\dot{X}(0)$ are the initial position and momentum operators. The solution (2.38) may be written in symmetrical form as

$$
\begin{aligned}
X(t) &= X(0)\cos t + \dot{X}(0)\sin t \\
&+ \tfrac{2\lambda}{768}X^5(0)(\cos t + 15\cos 3t - 16\cos t - 120t\sin t) \\
&+ \tfrac{\lambda}{768}[X^4(0)\dot{X}(0) + \dot{X}(0)X^4(0)] \\
&\times (5\sin 5t + 45\sin 3t - 280\sin t + 120t\cos t) \\
&+ \tfrac{2\lambda}{768}[X^3(0)\dot{X}^2(0) + \dot{X}^2(0)X^3(0) + 3X(0)] \\
&\times (-5\cos 5t - 15\cos 3t + 20\cos t - 120t\sin t) \\
&+ \tfrac{2\lambda}{768}[\dot{X}^3(0)X^2(0) + X^2(0)\dot{X}^3(0) + 3\dot{X}(0)] \\
&\times (-5\sin 5t + 15\sin 3t - 140\sin t + 120t\cos t) \\
&+ \tfrac{\lambda}{768}[X(0)\dot{X}^4(0) + \dot{X}^4(0)X(0)] \\
&\times (5\cos 5t - 45\cos 3t + 40\cos t - 120t\sin t) \\
&+ \tfrac{2\lambda\dot{X}^5(0)}{768}(\sin 5t - 15\sin 3t - 80\sin t + 120t\cos t).
\end{aligned} \tag{2.39}
$$

Equation (2.39) is our desired solution for a quantum sextic anharmonic oscillator. Now we rearrange our derived solution (2.39) with the help of (2.36) as

$$
\begin{aligned}
X(t) &= \tfrac{1}{2}\Big\{X(0)\cos\Big(t + \tfrac{5\lambda t}{4}\Big\{H_0^2 + \tfrac{1}{4}\Big\}\Big) + \cos\Big(t + \tfrac{5\lambda t}{4}\Big\{H_0^2 + \tfrac{1}{4}\Big\}\Big)X(0) \\
&+ \dot{X}(0)\sin\Big(t + \tfrac{5\lambda t}{4}\Big\{H_0^2 + \tfrac{1}{4}\Big\}\Big) + \sin\Big(t + \tfrac{5\lambda t}{4}\Big\{H_0^2 + \tfrac{1}{4}\Big\}\Big)\dot{X}(0) \\
&+ \tfrac{2\lambda X^5(0)}{384}(\cos 5t + 15\cos 3t - 16\cos t) \\
&+ \tfrac{\lambda}{384}[X^4(0)\dot{X}(0) + \dot{X}(0)X^4(0)](5\sin 5t + 45\sin 3t - 280\sin t) \\
&+ \tfrac{2\lambda}{384}[X^3(0)\dot{X}^2(0) + \dot{X}^2(0)X^3(0) + 3X(0)](-5\cos 5t - 15\cos 3t + 20\cos t) \\
&+ \tfrac{2\lambda}{384}[\dot{X}^3(0)X^2(0) + X^2(0)\dot{X}^3(0) + 3\dot{X}(0)](-5\sin 5t + 15\sin 3t - 140\sin t) \\
&+ \tfrac{\lambda}{384}[X(0)\dot{X}^4(0) + \dot{X}^4(0)X(0)](5\cos 5t - 45\cos 3t + 40\cos t) \\
&+ \tfrac{2\lambda\dot{X}^5(0)}{384}(\sin 5t - 15\sin 3t - 80\sin t)\Big\}
\end{aligned} \tag{2.40}
$$

where $H_0 = \frac{\dot{X}^2(0)}{2} + \frac{X^2(0)}{2}$ is the unperturbed Hamiltonian. Now the matrix element $< n-1|X(t)|n >$ and hence the frequency shift of the oscillator may be calculated with the help of (2.29). Now, the dipole matrix element in terms of the number eigenket $|n>$ is

$$
\begin{aligned}
< n-1|X(t)|n > &= \tfrac{\cos(\frac{5\lambda t}{4}n)}{2}\Big\{< n-1|X(0)|n > \cos\Big(t + \tfrac{5\lambda t}{4}\Big\{n^2 + \tfrac{1}{2}\Big\}\Big) \\
&+ \cos\Big(t + \tfrac{5\lambda t}{4}\Big\{n^2 + \tfrac{1}{2}\Big\}\Big) < n-1|X(0)|n > \\
&+ < n-1|\dot{X}(0)|n > \sin\Big(t + \tfrac{5\lambda t}{4}\Big\{n^2 + \tfrac{1}{2}\Big\}\Big) \\
&+ \sin\Big(t + \tfrac{5\lambda t}{4}\Big\{n^2 + \tfrac{1}{2}\Big\}\Big) < n-1|\dot{X}(0)|n > \\
&+ < n-1|\Big[\tfrac{2\lambda X^5(0)}{384}(\cos 5t + 15\cos 3t - 16\cos t) \\
&+ \tfrac{\lambda}{384}[X^4(0)\dot{X}(0) + \dot{X}(0)X^4(0)](5\sin 5t + 45\sin 3t - 280\sin t) \\
&+ \tfrac{2\lambda}{384}[X^3(0)\dot{X}^2(0) + \dot{X}^2(0)X^3(0) + 3X(0)](-5\cos 5t - 15\cos 3t + 20\cos t) \\
&+ \tfrac{2\lambda}{384}[\dot{X}^3(0)X^2(0) + X^2(0)\dot{X}^3(0) + 3\dot{X}(0)](-5\sin 5t + 15\sin 3t - 140\sin t) \\
&+ \tfrac{\lambda}{384}[X(0)\dot{X}^4(0) + \dot{X}^4(0)X(0)](5\cos 5t - 45\cos 3t + 40\cos t) \\
&+ \tfrac{2\lambda\dot{X}^5(0)}{384}(\sin 5t - 15\sin 3t - 80\sin t)\Big]|n >\Big\}.
\end{aligned} \tag{2.41}
$$

Hence, we obtain the frequency shift of the oscillator as $\frac{5\lambda}{4}(n^2 + \frac{1}{2})$. Note that the frequency shift of the oscillator is observed even when there are no photons (i.e vacuum field) present in the radiation field. The vacuum field interacts with the medium and causes the frequency shift. This phenomena is a direct outcome of pure quantum electrodynamic effect and has no classical analogue. The frequency shift of the oscillator for a vacuum field is actually a second order effect and is discussed in the context of the second order operator solution of the quartic oscillator. The solution (2.40) is consistent with the generalized solutions obtained in the later part of this thesis. At present, the frequency shift in (2.41) may be compared with the frequency shift calculated by using the first order perturbation technique [51]. As a check, we calculate the energy of a sextic oscillator in the $n-th$ eigenstate which is given by

$$E_n = (n + \frac{1}{2}) + \frac{5\lambda}{48}(4n^3 + 6n^2 + 8n + 3). \tag{2.42}$$

The energy difference between two consecutive levels is

$$\Delta E = E_n - E_{n-1} = 1 + \frac{5\lambda}{4}(n^2 + \frac{1}{2}). \tag{2.43}$$

Thus our calculated frequency shift and hence the solution agrees nicely. Finally we normalize the equation (2.40) with the help of the equation (2.41) as

$$\begin{aligned}
X(t) &= \tfrac{1}{2cos(\frac{5\lambda t}{4}n)}\left\{X(0)cos\left(t+\tfrac{5\lambda t}{4}\left\{H_0^2+\tfrac{1}{4}\right\}\right)+cos\left(t+\tfrac{5\lambda t}{4}\left\{H_0^2+\tfrac{1}{4}\right\}\right)X(0)\right.\\
&+ \dot{X}(0)sin\left(t+\tfrac{5\lambda t}{4}\left\{H_0^2+\tfrac{1}{4}\right\}\right)+sin\left(t+\tfrac{5\lambda t}{4}\left\{H_0^2+\tfrac{1}{4}\right\}\right)\dot{X}(0)\\
&+ \tfrac{2\lambda X^5(0)}{384}(cos5t + 15\,cos3t - 16\,cost)\\
&+ \tfrac{\lambda}{384}[X^4(0)\dot{X}(0)+\dot{X}(0)X^4(0)](5sin5t+45sin3t-280sint)\\
&+ \tfrac{2\lambda}{384}[X^3(0)\dot{X}^2(0)+\dot{X}^2(0)X^3(0)+3X(0)](-5cos5t-15cos3t+20cost)\\
&+ \tfrac{2\lambda}{384}[\dot{X}^3(0)X^2(0)+X^2(0)\dot{X}^3(0)+3\dot{X}(0)](-5sin5t+15sin3t-140sint)\\
&+ \tfrac{\lambda}{384}[X(0)\dot{X}^4(0)+\dot{X}^4(0)X(0)](5cos5t-45cos3t+40cost)\\
&+ \left.\tfrac{2\lambda\dot{X}^5(0)}{384}(sin5t-15sin3t-80sint)\right\}.
\end{aligned} \tag{2.44}$$

The equation (2.44) is the desired solution for quantum sextic anharmonic oscillator where the secular terms for all orders are summed up.

2.2.2 Classical and quantum solutions of the octic oscillator

Depending upon the nature of nonlinearity, the higher anharmonic oscillators come into the picture. For example, the Hamiltonian and the equation of motion of classical octic anharmonic oscillator may be obtained if we put $m = 8$ in the equations (2.5) and (2.6) respectively. Using the similar procedure (as adopted for sextic oscillator), the solution (up to the linear power of λ)

for classical octic oscillator follows immediately as

$$\begin{aligned}
x(t) &= x(0)cost + \dot{x}(0)sint \\
&+ \tfrac{\lambda x^7(0)}{3072}(cos7t + 14cos5t + 126cos3t - 141cost - 840tsint) \\
&+ \tfrac{\lambda \dot{x}(0)x^6(0)}{3072}(7sin7t + 70sin5t + 378sint - 2373sint + 840tcost) \\
&+ \tfrac{\lambda \dot{x}^2(0)x^5(0)}{3072}(-21cos7t - 126cos5t - 126cos3t + 273cost - 2520tsint) \\
&+ \tfrac{\lambda \dot{x}^3(0)x^4(0)}{3072}(-35sin7t - 70sin5t + 630sin3t - 3815sint + 2520tcost) \\
&+ \tfrac{\lambda \dot{x}^4(0)x^3(0)}{3072}(35cos7t - 70cos5t - 630cos3t + 665cost - 2520tsint) \\
&+ \tfrac{\lambda \dot{x}^5(0)x^2(0)}{3072}(21sin7t - 126sin5t + 126sin3t - 2415sint + 2520tcost) \\
&+ \tfrac{\lambda \dot{x}^6(0)x(0)}{3072}(-7cos7t + 70cos5t - 378cos3t + 315cost - 840tsint) \\
&+ \tfrac{\lambda \dot{x}^7(0)}{3072}(-sin7t + 14sin5t - 126sin3t - 525sint + 840tcost)
\end{aligned} \tag{2.45}$$

To check the validity of our solution, we consider a special case $x(0) = b$ and $\dot{x}(0) = 0$. After a little algebra we have

$$x(t) = bcos(1 + \frac{35\lambda}{128}b^6)t + \frac{\lambda b^7}{3072}(-141cost + 126cos3t + 14cos5t + cos7t). \tag{2.46}$$

Thus the frequency shift of the classical octic oscillator is proportional to the sixth power of amplitude as long as the first order solution is concerned. The solution (2.46) as well as the frequency shift have exact coincidence with the solution obtained by the procedure introduced by Bradbury and Brintzenhoff [55]. The solution of the classical octic oscillator (2.45) is now used to obtain the corresponding solution for its quantum counter part. It is found that the solution contains the secular terms similar to those appeared in (2.24). The problem of secular terms are taken care by the same procedure as it is done in the case of a sextic anharmonic oscillator and the final solution appears as

$$\begin{aligned}
X(t) &= \tfrac{1}{2cos[\frac{35\lambda}{64}(6n^2+3)t]}\Big\{X(0)cos\Omega t + cos\Omega t X(0) + \dot{X}(0)sin\Omega t + sin\Omega t\dot{X}(0) \\
&+ \tfrac{2\lambda X^7(0)}{3072}(cos7t + 14cos5t + 126cos3t - 141cost) \\
&+ \tfrac{\lambda}{3072}[\dot{X}(0)X^6(0) + X^6(0)\dot{X}(0)](7sin7t + 70sin5t + 378sint - 2373sint) \\
&+ \tfrac{\lambda}{3072}[\dot{X}^5(0)X^2(0) + X^2(0)\dot{X}^5(0) + 10X^3(0)] \\
&\times (-21cos7t - 126cos5t - 126cos3t + 273cost) \\
&+ \tfrac{\lambda}{3072}[\dot{X}^4(0)X^3(0) + X^3(0)\dot{X}^4(0) + 9(X^2(0)\dot{X}(0) + \dot{X}(0)X^2(0))] \\
&\times (-35sin7t - 70sin5t + 630sin3t - 3815sint) \\
&+ \tfrac{\lambda}{3072}[\dot{X}^3(0)X^4(0) + X^4(0)\dot{X}^3(0) + 9(\dot{X}^2(0)X(0) + X(0)\dot{X}^2(0))] \\
&\times (35cos7t - 70cos5t - 630cos3t + 665cost) \\
&+ \tfrac{\lambda}{3072}[\dot{X}^5(0)X^2(0) + X^2(0)\dot{X}^5(0) + 10\dot{X}^3(0)] \\
&\times (21sin7t - 126sin5t + 126sin3t - 2415sint) \\
&+ \tfrac{\lambda}{3072}[\dot{X}^6(0)X(0) + X(0)\dot{X}^6(0)](-7cos7t + 70cos5t - 378cos3t + 315cost) \\
&+ \tfrac{2\lambda \dot{X}^7(0)}{3072}(-sin7t + 14sin5t - 126sin3t - 525sint)\Big\}
\end{aligned} \tag{2.47}$$

where the first order frequency operator is $\Omega = 1 + \frac{35\lambda}{64}(4H_0^3 + 5H_0)$. Now, the matrix element $< n-1|X(t)|n >$ is calculated to obtain the shifted frequency as $\omega' = 1+\frac{35\lambda}{16}(n^3+2n)$. Interestingly, the frequency shift $\frac{35\lambda}{16}(n^3+2n)$ depends on n^3. It is clear that the vacuum field has no role to play in the frequency shift of the quantum octic oscillator. To have a possible verification of the solution (2.47), we calculate the frequency shift by using Rayleigh- Schrödinger perturbation theory. A first order calculation under this theory shows that the energy of the octic oscillator in the $n-th$ state is given by

$$E_n = (n + \frac{1}{2}) + \frac{35\lambda}{64}(\frac{3}{2} + 4n + 5n^2 + 2n^3 + n^4). \tag{2.48}$$

The energy difference between two consecutive levels is $\Delta E = E_n - E_{n-1} = 1 + \frac{35\lambda}{16}(n^3 + 2n)$. Thus, the frequency shift coincides exactly with the frequency shift obtained by us.

2.2.3 Remarks on the solutions:

We obtain analytical solutions (first order) for sextic (2.33) and octic (2.45) oscillators in classical pictures by using the Taylor series method. The solutions exhibit the presence of the usual secular terms. The secular terms for all orders are tucked in to obtain the frequency shifts of those oscillators. It is found that the calculated shifts agree satisfactorily with the shifts obtained by other methods.

The position and momentum of classical oscillators are replaced by their corresponding operators and the commutation relation is imposed. Hence, the solutions for quantum sextic and octic oscillators in the realm of the Heisenberg picture is obtained. Interestingly, the vacuum field has no role to play in the first order frequency shift of the quantum oscillators of quartic and octic anharmonicities. However, the vacuum field causes the frequency shift of a quantum sextic oscillator.

Till now we have discussed the Taylor series method for obtaining first order operator solution of the quartic, sextic and octic anharmonic oscillators. A close look into the difference equations and their solutions developed in the earlier sections shows some symmetries among them. Those symmetries encouraged us to search methods of obtaining solutions of $m-th$ anharmonic oscillator in general. In the next section of this chapter we develop a zeroth order MSPT solution for the $m-th$ anharmonic oscillator and generalize the first order operator solutions of quartic oscillator obtained by various other techniques to obtain the first order operator solution of generalized anharmonic oscillator in the subsequent sections. One more thing we would like to note here, the Taylor series and MSPT methods for obtaining solutions of (1.30) for particular values of m are difficult and tedious. So in the next few sections of this chapter we shall try to develop simpler techniques for solving (1.30) in general.

2.3 Generalized quantum anharmonic oscillator using an operator ordering approach

In the present section we study the generalized quantum anharmonic oscillator problem in the Heisenberg representation by making use of two simple normal ordering theorems for the expansion of $(a + a^\dagger)^m$. We observe that the merit of the present work is its simplicity. For example, the present method provides, on the one hand, a straight forward mathematical frame work to construct expressions for the energy eigenvalues as well as frequency shifts and, on the other hand, generalizes some of the recent results on the topic by Bender and Bettencourt [30-31]. In the subsequent sections of these chapter we have shown that the above mentioned theorems may be used to generalize the results obtained by other methods.

We have already mentioned that the equation of motion corresponding to the Hamiltonian (1.29) of a generalized anharmonic oscillator having unit mass and unit frequency is given by (1.30) which can not be solved exactly for $m > 2$. However, a large number of approximation methods are available for solving (1.30) for particular values of m. We have discussed the Taylor series method in detail and have mentioned some other methods. MSPT is one of those methods. Bender and Bettencourt [30,31] generalized the existing theory of MSPT into an operator approach to get the zeroth order solution involving a quantum operator analogue of the classical first order frequency shift. Following the interpretation of Aks [28,29] they interpreted it as an operator mass renormalization that expresses the first order shift of all energy levels. They obtained MSPT operator solution of a quantum quartic oscillator. In the present work we suggest a novel method in which MSPT results for anharmonic oscillators in general can be found. To that end we prove in the next subsection two theorems to construct a normal ordered expansion of $(a + a^\dagger)^m$. In the subsequent subsections we obtain a generalized expression for the energy eigenvalues and a generalized solution for the equation of motion (1.30). We also specialize our result to reproduce some of the existing results as useful checks on the generalized solution obtained by us.

2.3.1 Operator ordering theorems

On a very general ground one knows that in quantum mechanics and quantum field theory proper ordering of the operators plays a crucial role. Let $f(a, a^\dagger)$ be an arbitrary operator function of the usual bosonic annihilation and creation operators a and $a^\dagger$ which satisfy the commutation relation

$$[a, a^\dagger] = 1. \tag{2.49}$$

One can write $f(a, a^\dagger)$ in such a way that all powers of $a^\dagger$ always appear to the left of all powers of a. Then $f(a, a^\dagger)$ is said to be normal ordered. In this work we want to write $(a + a^\dagger)^m$ in the normal ordered form for integral values of m. Traditionally, for a given value of m this is achieved by using a very lengthy procedure which involves repeated application of (2.49). One of our objectives in this work is to construct a normal ordered expansion of $(a + a^\dagger)^m$ without taking recourse to such repeated applications.

We denote normal ordered form of f by f_N. On the other hand $:\ f\ :$ denote an operator obtained from f by arranging all powers of $a^\dagger$ to the left of all powers of a without making use of the commutation relation (2.49). Now if $f = aa^\dagger$ then $f_N = a^\dagger a + 1$ and $:\ f\ := a^\dagger a$. Therefore, we can write

$$:\ (a + a^\dagger)^m :=:\ (a^\dagger + a)^m := a^m +^m C_1 a^\dagger a^{m-1} + ... +^m C_r a^{\dagger^r} a^{m-r} + .. + a^{\dagger^m}. \tag{2.50}$$

Thus in this notation $:\ (a + a^\dagger)^m\ :$ is simply a binomial expansion in which powers of the $a^\dagger$ are always kept to the left of the powers of the a. To write $(a^\dagger + a)^m_N$ we can proceed by using the following theorems.

2.3.1.1 Theorem 1.

$$:\ (a^\dagger + a)^m :\ (a^\dagger + a) =:\ (a^\dagger + a)^{m+1} : +m\ :\ (a^\dagger + a)^{m-1} :\ . \tag{2.51}$$

Proof: From (2.50) $:\,(a^\dagger+a)^m\,:\,(a^\dagger+a)$ can be written in the form

$$
\begin{aligned}
& :\,(a^\dagger+a)^m\,:\,(a^\dagger+a) \\
= \; & \left[a^{\dagger^{m+1}} + ({}^mC_1 +{}^m C_0) a^{\dagger^m} a + ({}^mC_2 +{}^m C_1) a^{\dagger^{m-1}} a^2 + ... + ({}^mC_r +{}^m C_{r-1}) a^{\dagger^{m-r+1}} a^r + ...\right] \\
+ \; & ({}^mC_1 a^{\dagger^{m-1}} + 2\,{}^mC_2 a^{\dagger^{m-2}} a + ... + r\,{}^mC_r a^{\dagger^{m-r}} a^{r-1} + ...) \\
= \; & :\,(a^\dagger+a)^{m+1}\,: + m\,:\,(a^\dagger+a)^{m-1}\,:\ .
\end{aligned} \tag{2.52}
$$

In the above we have made use of the identities

$$
\begin{aligned}
(a^r a^\dagger)_N &= a^\dagger a^r + r a^{r-1}, \\
r\,{}^nC_r &= n\,{}^{n-1}C_{r-1}, \\
and\ \ {}^{m+1}C_{r+1} &= ({}^mC_r +{}^m C_{r+1})\,.
\end{aligned} \tag{2.53}
$$

Note that first identity in (2.53) can be proved with the help of the general operator ordering theorems [56] while the other two are trivial.

2.3.1.2 Theorem 2 :

For any integral values of m

$$(a^\dagger+a)_N^m = \sum_{r=0,2,4..}^{m} t_r\,{}^mC_r\,:\,(a^\dagger+a)^{m-r}\,: \tag{2.54}$$

with

$$t_r = \frac{r!}{2^{\frac{r}{2}}(\frac{r}{2})!} \tag{2.55}$$

Proof[2]:
Using theorem 1, we can write

$$
\begin{aligned}
(a^\dagger+a)_N &= \;:\,(a^\dagger+a)\,: \\
(a^\dagger+a)_N^2 &= \;:\,(a^\dagger+a)^2\,: + 1 \\
(a^\dagger+a)_N^3 &= \;:\,(a^\dagger+a)^3\,: + 3\,:\,(a^\dagger+a)\,: \\
(a^\dagger+a)_N^4 &= \;:\,(a^\dagger+a)^4\,: + 6\,:\,(a^\dagger+a)^2\,: + 3 \\
(a^\dagger+a)_N^5 &= \;:\,(a^\dagger+a)^5\,: + 10\,:\,(a^\dagger+a)^3\,: + 15\,:\,(a^\dagger+a)\,: \\
(a^\dagger+a)_N^6 &= \;:\,(a^\dagger+a)^6\,: + 15\,:\,(a^\dagger+a)^4\,: + 45\,:\,(a^\dagger+a)^2\,: + 15 \\
(a^\dagger+a)_N^7 &= \;:\,(a^\dagger+a)^7\,: + 21\,:\,(a^\dagger+a)^5\,: + 105\,:\,(a^\dagger+a)^3\,: \\
&+ \;105\,:\,(a^\dagger+a)\,: \\
(a^\dagger+a)_N^8 &= \;:\,(a^\dagger+a)^8\,: + 28\,:\,(a^\dagger+a)^6\,: + 210\,:\,(a^\dagger+a)^4\,: \\
&+ \;420\,:\,(a^\dagger+a)^2\,: + 105 \\
(a^\dagger+a)_N^9 &= \;:\,(a^\dagger+a)^9\,: + 36\,:\,(a^\dagger+a)^7\,: + 378\,:\,(a^\dagger+a)^5\,: \\
&+ \;1260\,:\,(a^\dagger+a)^3\,: + 945\,:\,(a^\dagger+a)\,:\ .
\end{aligned} \tag{2.56}
$$

From (2.56) we venture to identify the general form of the above expansion for a given value of m as

$$(a^\dagger+a)_N^m = \sum_{r=0,2,4..}^{m} t_r\,{}^mC_r\,:\,(a^\dagger+a)^{m-r}\,:\ . \tag{2.57}$$

[2]F M Fernandez [40] has pointed out that this theorem can be proved by expanding both sides of the Campbell-Baker-Hausdroff formula [56]

$$e^{\xi(a+a^\dagger)} = e^{\xi a^\dagger} e^{\xi a} e^{\xi^2}$$

about $\xi = 0$.

It is easy to check that (2.57) gives all the expansions of (2.56). So (2.57) is true for $m = 1, 2, .., 9$. To ensure the general validity of (2.57) we can use the method of induction. First we assume that (2.57) is true for a particular value of m. Now,

$$\begin{aligned}(a^\dagger + a)^{m+1}_N &= (a^\dagger + a)^m_N\,(a^\dagger + a)\\ &= \Big[:\,(a^\dagger + a)^m : + {}^mC_2 :\,(a^\dagger + a)^{m-2} : + ...\\ &+\ t_{r-2}\,{}^mC_{r-2} : (a^\dagger + a)^{m-r+2} : + t_r\,{}^mC_r :\,(a^\dagger + a)^{m-r} : + ...\Big]\,(a^\dagger + a)\\ &= \textstyle\sum_r \left[t_{r-2}\,{}^mC_{r-2}(m - r + 2) + t_r\,{}^mC_r\right]\,:\,(a^\dagger + a)^{m-r+1} :\\ &= \textstyle\sum_r t_r\,{}^{m+1}C_r :\,(a^\dagger + a)^{m+1-r} :\ .\end{aligned} \tag{2.58}$$

Equation (2.58) exhibits that if (2.57) holds for any arbitrary m it must hold for $m + 1$. This establishes that (2.57) gives the normal order expansion of $(a^\dagger + a)^m$ for any arbitrary integer m. We use this normal ordered expansion to study the anharmonic oscillator problem.

2.3.2 Energy eigenvalues

The first order energy eigenvalue E_1 is $< n|H|n >$, where $|n >$ is the number state. In the literature there are two lengthy procedures [51] to obtain E_1. The first one is the usual normal ordering method. This method involves iterative use of (2.49) and the number of iterations increases very fast as m increases. For a given value of m we need $[2^m - (m + 1)]$ iterations. Just to get a feeling of the number of iterations required for large values of m we can look at table 1 below.

m	no. of iterations required
2	1
6	57
10	1013
20	1048555
100	1267650600228229401496703205275

Table 2.1

The table shows that for large m the construction of the expansion $(a^\dagger + a)^m_N$ becomes formidable. In the second procedure one proceeds by making repeated application of x operator on the number state. As in the normal ordering method this procedure is also equally lengthy. In the following we implement theorem 2 to derive an uncomplicated method to construct an expansion for E_1 for the Hamiltonian (1.29).

Using the result in (2.54) we can write the expression for E_1 for any integer value of m in the form

$$\begin{aligned}E_1 &= (n + \tfrac{1}{2}) + \tfrac{\lambda}{2^{\frac{m}{2}}m} < n| \textstyle\sum_{r=0,2,4..}^m t_r\,{}^mC_r :\,(a^\dagger + a)^{m-r} :\,|n >\\ &= (n + \tfrac{1}{2}) + \tfrac{\lambda}{2^{\frac{m}{2}}m} < n| \textstyle\sum_{r=0,2,4..}^m t_r\,{}^mC_r\,{}^{m-r}C_{\frac{m-r}{2}}\,a^{\dagger\frac{m-r}{2}}\,a^{\frac{m-r}{2}}|n >\\ &= (n + \tfrac{1}{2}) + \tfrac{\lambda}{2^{\frac{m}{2}}m} \textstyle\sum_{r=0,2,4..}^m t_r\,{}^mC_r\,{}^{m-r}C_{\frac{m-r}{2}}\,{}^nC_{\frac{m-r}{2}}\,(\frac{m-r}{2})!\end{aligned} \tag{2.59}$$

The expression (2.59) involves summations which are easy to evaluate and thereby avoids the difficulties associated with earlier iterative procedures. Although somewhat forced, it may be tempting to compare the simplicity sought in our approach with the use of logarithm in a numerical calculation or use of integral transforms in solving a partial differential equation. It is of interest to note that expression similar to (2.59) can also be constructed for the first order energy eigenvalue of a Hamiltonian in which the anharmonic term is a polynomial in X.

2.3.3 MSPT solution of the generalized quantum anharmonic oscillator

We want to obtain MSPT operator solution of equation (1.30) for arbitrary integral values of m. The essential idea behind our approach is that the quantum operator analogue of the classical first order frequency shift is an operator function ($\Omega(H_0)$) of the unperturbed Hamiltonian H_0 and secondly a correct solution should reproduce the first order energy spectrum. Now the general form of the zeroth order solution of (1.30) is given by

$$\begin{aligned} X_0(t) &= \frac{1}{G(n)}\left[X(0)cos(t+\lambda\Omega(H_0)t) + cos(t+\lambda\Omega(H_0)t)X(0)\right. \\ &+ \left.\dot{X}(0)sin(t+\lambda\Omega(H_0)t) + sin(t+\lambda\Omega(H_0)t)\dot{X}(0)\right] \end{aligned} \tag{2.60}$$

where $G(n)$ is a normalization factor. So our task is to find out $\Omega(H_0)$ and $G(n)$ in general. From equation (2.59) we obtain the energy difference for two consecutive energy levels as

$$\begin{aligned} \omega_{n,n-1} &= (E_1)_{m,n} - (E_1)_{m,n-1} \\ &= 1 + \lambda\omega(m,n) \\ &= 1 + \frac{\lambda}{2^{\frac{m}{2}}m}\sum_{r=0,2,4..}^{(m-2)} t_r\, {}^mC_r\, {}^{m-r}C_{\frac{m-r}{2}}\, {}^{n-1}C_{\frac{m-r-2}{2}}\, (\tfrac{m-r}{2})!\,. \end{aligned} \tag{2.61}$$

Since the correct quantum operator solution has to give the first order energy spectrum, we should have

$$\begin{aligned} < n-1|X_0(t)|n > &= < n-1|X_0(0)|n > cos[t+\lambda\omega(m,n)t] \\ &+ < n-1|\dot{X}_0(0)|n > sin[t+\lambda\omega(m,n)t]\,. \end{aligned} \tag{2.62}$$

Equation (2.60) and (2.62) impose restrictions on our unknown functions $\Omega(H_0)$ and $G(n)$. The condition imposed on $\Omega(H_0)$ is

$$< n|\Omega(H_0)|n > + < n-1|\Omega(H_0)|n-1 >= 2\omega(m,n) \tag{2.63}$$

or,

$$\Omega(n+\frac{1}{2}) + \Omega(n-\frac{1}{2}) = 2\omega(m,n)\ . \tag{2.64}$$

For a particular m the right hand side is a known polynomial in n and our job is simply to find out $\Omega(n+\frac{1}{2})$. We obtain this as

$$\Omega(n+\frac{1}{2}) = 2\left[\sum_{k=0}^{n}(-1)^{n-k}\omega(m,k)\right] + (-1)^{n+\frac{m}{2}}\frac{t_m}{2^{\frac{m-2}{2}}m}. \tag{2.65}$$

Substituting the functional form of $\Omega(n+\frac{1}{2})$ or $\Omega(H_0)$ in (2.60) if we impose condition (2.62) we get

$$G(n) = 2cos\left[\frac{\lambda t}{2}\left(\Omega(n+\frac{1}{2}) - \Omega(n-\frac{1}{2})\right)\right]. \tag{2.66}$$

The results in (2.65) and (2.66) when substituted in (2.60) solve the generalized quantum anharmonic oscillator problem.

2.3.4 Specific results and their comparison with the existing spectra:

Although we have solved the quantum anharmonic oscillator in general, just now it is not possible to compare our solution directly with any other result since the present study happens to be the first operator solution of the generalized anharmonic oscillator. Later on we develop other techniques to solve the anharmonic oscillator in general and compare those result with the present one. Here we calculate some specific results from our general expressions and compare them with the existing results.

For $m = 4$ we have,

$$\Omega(n+\frac{1}{2}) = \frac{3n}{4} + \frac{3}{8} = \frac{3}{4}(n+\frac{1}{2})\,. \tag{2.67}$$

Therefore,

$$\begin{array}{rcl} \Omega(H_0) & = & \frac{3}{4}H_0 \\ and \quad G(n) & = & 2cos(\frac{3\lambda t}{8})\,. \end{array} \tag{2.68}$$

In terms of (2.68) the total solution for the quantum quartic anharmonic oscillator is

$$\begin{array}{rcl} X(t)|_{m=4} & = & \frac{1}{2cos(\frac{3\lambda t}{8})}\Big[X(0)cos[t+\frac{3\lambda t}{4}H_0] + cos[t+\frac{3\lambda t}{4}H_0]X(0) \\ & + & \dot{X}(0)sin[t+\frac{3\lambda t}{4}H_0] + sin[t+\frac{3\lambda t}{4}H_0]\dot{X}(0)\Big] \end{array} \tag{2.69}$$

This coincides exactly with the solution given by Bender and Bettencourt [30,31] and also coincides with the Taylor series solution (2.26). This also gives the correct classical frequency shift [57] in the limit $x(0) = a,\ \dot{x}(0) = 0$. Similarly we have

$$\begin{array}{rcl} X(t)|_{m=6} & = & \frac{1}{2cos(\frac{5\lambda t}{4}n)}\Big[X(0)cos[t+\frac{5\lambda}{4}(H_0^2+\frac{1}{4})t] + cos[t+\frac{5\lambda}{4}(H_0^2+\frac{1}{4})t]X(0) \\ & + & \dot{X}(0)sin[t+\frac{5\lambda}{4}(H_0^2+\frac{1}{4})t] + sin[t+\frac{5\lambda}{4}(H_0^2+\frac{1}{4})t]\dot{X}(0)\Big] \end{array}, \tag{2.70}$$

$$\begin{array}{rcl} x(t)|_{m=8} & = & \frac{1}{2cos[\frac{35\lambda}{64}(6n^2+3)t]} \\ & \times & \Big[X(0)cos[t+\frac{35\lambda}{64}(4H_0^3+5H_0)t] + cos[t+\frac{35\lambda}{64}(4H_0^3+5H_0)t]X(0) \\ & + & \dot{X}(0)sin[t+\frac{35\lambda}{64}(4H_0^3+5H_0)t] + sin[t+\frac{35\lambda}{64}(4H_0^3+5H_0)t]\dot{X}(0)\Big]\,, \end{array}$$

and

$$\begin{array}{rcl} x(t)|_{m=10} & = & \frac{1}{2cos[\frac{63\lambda t}{8}(n^3+2n)]}\Big[X(0)cos[t+\frac{63\lambda}{16}(H_0^4+\frac{7}{2}H_0^2+\frac{9}{16})t] \\ & + & cos[t+\frac{63\lambda}{16}(H_0^4+\frac{7}{2}H_0^2+\frac{9}{16})t]X(0) + \dot{X}(0)sin[t+\frac{63\lambda}{16}(H_0^4+\frac{7}{2}H_0^2+\frac{9}{16})t] \\ & + & sin[t+\frac{35\lambda}{64}\frac{63\lambda}{16}(H_0^4+\frac{7}{2}H_0^2+\frac{9}{16})t]\dot{X}(0)\Big]\,. \end{array} \tag{2.71}$$

The solutions for sextic and octic oscillator exactly coincides with the solution obtained by us using Taylor series approach [39] and all solutions give the correct classical frequency shifts in the appropriate limit [54, 55].

2.3.5 Remarks on the solutions

We conclude by noting that depending on the nature of nonlinearity in a physical problem the treatment of higher anharmonic oscillators assumes significance. But studies in such oscillators (for $m > 4$) are not undertaken in the Heisenberg approach presumably because the existing methods tends to introduce inordinate mathematical complications in a detailed study. In the present work we contemplate to circumvent them by proving a theorem for the expansion of

$(a^\dagger + a)_N^m$. Thus the results of the present section are expected to serve a useful purpose for physicists working in nonlinear mechanics, molecular physics, quantum optics and quantum field theory. In fact, we use these solutions to study nonlinear and quantum optical effects in detail in the later part of the present thesis.

2.4 Generalized quantum anharmonic oscillator using the time evolution operator approach

Generalized quantum anharmonic oscillator can be studied through some alternative approaches. We have started our discussion on the possible alternative approaches from the MSPT approach. In the present section we study the time evolution operator approach. In this approach we find out the perturbed time evolution operator in the interaction picture $U_I(t)$ as a function of $a, a^\dagger$ and t and then we use that to find out the time evolution of the creation and annihilation operators which in turn solve the equation of motion (1.30) of the generalized anharmonic oscillator. The secular terms are removed from the solution by using the tucking in technique, discussed in the context of Taylor series solution.

2.4.1 Time evolution operator in the interaction picture:

In the interaction picture potential V_I and the time evolution operator $U_I(t)$ are respectively

$$V_I(t) = \exp(iH_0t)V\exp(-iH_0t) \tag{2.72}$$

and

$$U_I(t) = 1 - i\int_0^t dt_1 V_I(t_1) + (-i)^2\int_0^t dt_1 V_I(t_1)\int_0^{t_1} dt_2 V_I(t_2) + \ldots\ . \tag{2.73}$$

Where the suffix I stands for the interaction picture. Now to obtain the compact expressions for $V_I(t)$ and $U_I(t)$ (up to first order) for the model Hamiltonian (6.16) of our interest we have to use following operator ordering theorems

Theorem 3: When $f\left(a, a^\dagger\right)$ can be expanded as a power series in a and $a^\dagger$ then

$$\exp(\varsigma a^\dagger a) f\left(a, a^\dagger\right)\exp(-\varsigma a^\dagger a) = f\left(a\exp(-\varsigma), a^\dagger\exp(\varsigma)\right) \tag{2.74}$$

where ς is a c-number [56].

With the help of theorem 2 (see section 2.3.1.2) and theorem 3 we can write

$$\begin{array}{rcl} V_I(t) & = & \exp(ia^\dagger at)\frac{\lambda}{m(2)^{\frac{m}{2}}}(a^\dagger + a)^m\exp(-ia^\dagger at) \\ & = & \frac{\lambda}{m(2)^{\frac{m}{2}}}\sum_r^{\frac{m}{2}} t_{2r}\ {}^mC_{2r}\sum_p^{(m-2r)}\ {}^{(m-2r)}C_p a^{\dagger p}a^{m-2r-p}\exp\left(it(2p-m+2r)\right) \end{array} . \tag{2.75}$$

Now the first order time evolution operator in the interaction picture is

$$\begin{aligned}
U_I(t) &= 1 - i\int_0^t dt_1 V_I(t_1) \\
&= 1 - i\frac{\lambda}{m(2)^{\frac{m}{2}}}\int_0^t \left(\sum_r^{\frac{m}{2}} t_{2r}\, {}^mC_{2r} \sum_p^{(m-2r)}\, {}^{(m-2r)}C_p a^{\dagger^p} a^{m-2r-p} \exp\left(it_1(2p-m+2r)\right)\right) dt_1 \\
&= 1 - \frac{\lambda}{m(2)^{\frac{m}{2}}}\left(\sum_r^{\frac{m}{2}} t_{2r}\, {}^mC_{2r} \sum_{p\neq\frac{m-2r}{2}}^{(m-2r)}\, {}^{(m-2r)}C_p a^{\dagger^p} a^{m-2r-p} \frac{[\exp(it(2p-m+2r))-1]}{2p-m+2r}\right) \\
&- i\frac{\lambda t}{m(2)^{\frac{m}{2}}}\left(\sum_r^{\frac{m}{2}} t_{2r}\, {}^mC_{2r}\, {}^{(m-2r)}C_{\frac{m-2r}{2}} a^{\dagger^{\frac{m-2r}{2}}} a^{\frac{m-2r}{2}}\right).
\end{aligned} \tag{2.76}$$

The time evolution of the annihilation operator a in the Heisenberg picture a_H and in the interaction picture a_I are related by

$$a_H(t) = U_I^\dagger(t)\exp(ia^\dagger at)a(0)\exp(-ia^\dagger at)U_I(t) = \exp(-it)a_I(t). \tag{2.77}$$

Now the time evolution of the annihilation operator in the interaction picture is

$$\begin{aligned}
a_I(t) &= U_I^\dagger(t)a(0)U_I(t) \\
&= a - \frac{\lambda}{m(2)^{\frac{m}{2}}}\left(\sum_r^{\frac{m}{2}} t_{2r}\, {}^mC_{2r} \sum_{p\neq\frac{m-2r}{2}}^{(m-2r)}\, {}^{(m-2r)}C_p a^{\dagger^{p-1}} a^{m-2r-p} \frac{p[\exp(it(2p-m+2r))-1]}{2p-m+2r}\right) \\
&- i\frac{\lambda t}{m(2)^{\frac{m}{2}}}\left(\sum_r^{\frac{m}{2}} \frac{m-2r}{2} t_{2r}\, {}^mC_{2r}\, {}^{(m-2r)}C_{\frac{m-2r}{2}} a^{\dagger^{\frac{m-2r}{2}-1}} a^{\frac{m-2r}{2}}\right).
\end{aligned} \tag{2.78}$$

So we have obtained the time evolution of the annihilation operator.

2.4.2 First order frequency operator:

In the solution of (1.30) frequency becomes a operator function. In the first order frequency operator is a function of the unperturbed Hamiltonian H_0. The frequency operator is known for various values of m. Here we give a compact expression for the first order frequency operator $\Omega(H_0)$ and write the expression for some particular m to compare our result with the existing results. To do so we have to use the following identity

$$\begin{aligned}
a^{\dagger^n} a^n &= a^\dagger a(a^\dagger a - 1)(a^\dagger a - 2)...(a^\dagger a - n + 1) \\
&= (H_0 - \tfrac{1}{2})(H_0 - \tfrac{3}{2})....(H_0 - n + \tfrac{1}{2}) \\
&= \frac{\Gamma(H_0 + \frac{1}{2})}{\Gamma(H_0 - n + \frac{1}{2})}.
\end{aligned} \tag{2.79}$$

The secular terms present in (2.78) can be tucked in into the expression of a by using the tucking in technique introduced in (2.21). Now using (2.78 and 2.36) we can write

$$\begin{aligned}
a(t) &= \exp(-i\lambda\Omega t)a \\
&- \frac{\lambda}{m(2)^{\frac{m}{2}}}\left(\sum_r^{\frac{m}{2}} t_{2r}\, {}^mC_{2r} \sum_{p\neq\frac{m-2r}{2}}^{(m-2r)}\, {}^{(m-2r)}C_p a^{\dagger^{p-1}} \exp(i\Omega t(2p-m+2r-1))\right. \\
&\times \left. a^{m-2r-p} \frac{p[\exp(it(2p-m+2r))-1]}{2p-m+2r}\right)
\end{aligned} \tag{2.80}$$

where

$$\begin{aligned}
\Omega &= \frac{1}{m(2)^{\frac{m}{2}}}\left(\sum_r^{\frac{m}{2}} \frac{m-2r}{2} t_{2r}\, {}^mC_{2r}\, {}^{(m-2r)}C_{\frac{m-2r}{2}} a^{\dagger^{\frac{m-2r}{2}-1}} a^{\frac{m-2r}{2}-1}\right) \\
&= \frac{1}{m(2)^{\frac{m}{2}}}\left(\sum_r^{\frac{m}{2}} \frac{m-2r}{2} t_{2r}\, {}^mC_{2r}\, {}^{(m-2r)}C_{\frac{m-2r}{2}} \frac{\Gamma(H_0+\frac{1}{2})}{\Gamma(H_0-\frac{m-2r}{2}+\frac{3}{2})}\right)
\end{aligned}. \tag{2.81}$$

Thus we have a compact expression for the first order correction to the frequency operator (2.81) and an expression for the time evolution of annihilation operator which is free from secular term (2.80). Expression (2.80) essentially solves the equation of motion (1.30) of the generalized anharmonic oscillator since

$$X(t) = \frac{1}{\sqrt{2}}(a^\dagger(t) + a(t)). \tag{2.82}$$

Here we must note that the frequency operator of the annihilation operator is not exactly the same with the operator usually used to write the first order frequency correction of the position operator. Relationship and equivalence criterion for the frequency operators obtained by using different techniques are discussed at the end of this chapter.

2.5 Generalized quantum anharmonic oscillator using renormalization group technique

In this method a time parameter, additional to the initial value point is introduced, in such a way that the perturbation expansion is valid in the vicinity of the introduced time parameter. The coupling constants, constants of motion and/or initial conditions are suitably changed by the introduced time parameter. But the solution should not depend on the introduced time parameter so the first derivative of the perturbed solution with respect to introduced time parameter should be zero and this condition is known as RG condition. Solution of the differential equation imposed by RG condition can give provide us an operator solution of the anharmonic oscillator free from secular terms.

From the perturbation expansion (2.78) the secular part of the annihilation operator up to the first order is

$$a_{sec} = -i\frac{\lambda(t-\tau)}{m(2)^{\frac{m}{2}}}\left(\sum_r^{\frac{m}{2}} \frac{m-2r}{2} t_{2r}\, {}^mC_{2r}\, {}^{(m-2r)}C_{\frac{m-2r}{2}} a^{\dagger^{\frac{m-2r}{2}-1}} a^{\frac{m-2r}{2}}\right). \tag{2.83}$$

Now imposing the RG condition $\frac{da}{d\tau} = 0$, we obtain

$$\frac{da}{d\tau} = -i\frac{\lambda}{m(2)^{\frac{m}{2}}}\left(\sum_r^{\frac{m}{2}} \frac{m-2r}{2} t_{2r}\, {}^mC_{2r}\, {}^{(m-2r)}C_{\frac{m-2r}{2}} a^{\dagger^{\frac{m-2r}{2}-1}} a^{\frac{m-2r}{2}}\right). \tag{2.84}$$

Since $a^\dagger a$ and $[a, a^\dagger]$ are constants under the flow of τ, we are allowed to solve equation (2.84) as

$$a(\tau) = \exp\left(-i\frac{\lambda\tau}{m(2)^{\frac{m}{2}}}\left(\sum_r^{\frac{m}{2}} \frac{m-2r}{2} t_{2r}\, {}^mC_{2r}\, {}^{(m-2r)}C_{\frac{m-2r}{2}} a^{\dagger^{\frac{m-2r}{2}-1}} a^{\frac{m-2r}{2}-1}\right)\right) a(0). \tag{2.85}$$

Therefore the first order correction to frequency operator is

$$\begin{aligned}\Omega_1 &= \frac{1}{m(2)^{\frac{m}{2}}}\left(\sum_r^{\frac{m}{2}} \frac{m-2r}{2} t_{2r}\, {}^mC_{2r}\, {}^{(m-2r)}C_{\frac{m-2r}{2}} a^{\dagger^{\frac{m-2r}{2}-1}} a^{\frac{m-2r}{2}-1}\right) \\ &= \frac{1}{m(2)^{\frac{m}{2}}}\left(\sum_r^{\frac{m}{2}} \frac{m-2r}{2} t_{2r}\, {}^mC_{2r}\, {}^{(m-2r)}C_{\frac{m-2r}{2}} \frac{\Gamma(H_0+\frac{1}{2})}{\Gamma(H_0-\frac{m-2r}{2}+\frac{3}{2})}\right)\end{aligned}. \tag{2.86}$$

This is in exact accordance with the first order frequency operator derived by the time evolution operator method (2.81).

2.6 Generalized quantum anharmonic oscillator using near-identity transformation technique

The near-identity transformation relates the full solution for the annihilation operator a to the zeroth order term b as

$$a(t) = b(t) + \lambda T_1(b^\dagger(t), b(t)) + \lambda^2 T_2(b^\dagger(t), b(t)) + ... \tag{2.87}$$

and the equation for the time dependence of the zeroth order term in the normal form is

$$\frac{db(t)}{dt} = U_0(b^\dagger(t), b(t)) + \lambda U_1(b^\dagger(t), b(t)) + \lambda^2 U_2(b^\dagger(t), b(t)) + ... \tag{2.88}$$

Now using the Heisenberg equation of motion for the Hamiltonian (1.29) we obtain

$$\begin{aligned} \frac{da}{dt} &= ia + i\frac{\lambda}{2^{\frac{m}{2}}}(a + a^\dagger)^{m-1} \\ &= ia + i\frac{\lambda}{2^{\frac{m}{2}}}\sum_{r=0}^{\frac{m-1}{2}} t_{2r}\, {}^{m-1}C_{2r}\, {}^{(m-2r-1)}C_p a^{\dagger^p} a^{m-2r-1-p} \end{aligned} . \tag{2.89}$$

Inserting equation (2.87 and 2.88) into (2.89) we obtain a relation between U_n and T_n in each order n. These relations can not determine U_n and T_n uniquely. To reduce this nonuniqueness problem we follow the work of Kahn and Zarmi [34] and choose T_n not to depend on time explicitly. In the first order only the resonant terms of the form $(b^\dagger b)^k b$ acts as the source terms for the explicitly time dependent part of the annihilation operator a. Thus from the expansion (2.89) we have

$$U_1 = \frac{1}{2^{\frac{m}{2}}}\sum_{r=0}^{(\frac{m}{2}-1)} t_{2r}\, {}^{m-1}C_{2r}\, {}^{(m-2r-1)}C_{\frac{m-2r}{2}-1} b^{\dagger^{\frac{m-2r}{2}-1}}(t) b^{\frac{m-2r}{2}}(t). \tag{2.90}$$

Substituting (2.90) in (2.88) we obtain the dynamical equation of $b(t)$ as

$$\frac{db(t)}{dt} = -ib(t) + \frac{\lambda}{2^{\frac{m}{2}}}\sum_{r=0}^{(\frac{m}{2}-1)} t_{2r}\, {}^{m-1}C_{2r}\, {}^{(m-2r-1)}C_{\frac{m-2r}{2}-1} b^{\dagger^{\frac{m-2r}{2}-1}}(t) b^{\frac{m-2r}{2}}(t) .$$

Therefore we have

$$b = \exp\left(-it\left(1 + \frac{\lambda}{2^{\frac{m}{2}}}\sum_{r=0}^{(\frac{m}{2}-1)} t_{2r}\, {}^{m-1}C_{2r}\, {}^{(m-2r-1)}C_{\frac{m-2r}{2}-1} b_0^{\dagger^{\frac{m-2r}{2}-1}} b_0^{\frac{m-2r}{2}-1}\right)\right) b_0. \tag{2.91}$$

Now since the perturbation starts at $t = 0$ so $b(0) = a(0) + \mathcal{O}(\lambda)$ and the zeroth order term b of the annihilation operator is

$$b(t) = \exp\left(-it\left(1 + \frac{\lambda}{2^{\frac{m}{2}}}\sum_{r=0}^{(\frac{m}{2}-1)} t_{2r}\, {}^{m-1}C_{2r}\, {}^{(m-2r-1)}C_{\frac{m-2r}{2}-1} a^{\dagger^{\frac{m-2r}{2}-1}} a^{\frac{m-2r}{2}-1}\right)\right) a(0).$$

Therefore the first order correction to the frequency operator is

$$\begin{aligned} \Omega_1 &= \frac{1}{2^{\frac{m}{2}}}\sum_{r=0}^{(\frac{m}{2}-1)} t_{2r}\, {}^{m-1}C_{2r}\, {}^{(m-2r-1)}C_{\frac{m-2r}{2}-1} a^{\dagger^{\frac{m-2r}{2}-1}} a^{\frac{m-2r}{2}-1} \\ &= \frac{1}{2^{\frac{m}{2}}}\sum_{r=0}^{(\frac{m}{2}-1)} t_{2r}\, {}^{m-1}C_{2r}\, {}^{(m-2r-1)}C_{\frac{m-2r}{2}-1}\frac{\Gamma(H_0+\frac{1}{2})}{\Gamma(H_0-\frac{m-2r}{2}+\frac{3}{2})} \\ &= \frac{1}{m(2)^{\frac{m}{2}}}\left(\sum_{r}^{\frac{m}{2}} \frac{m-2r}{2} t_{2r}\, {}^{m}C_{2r}\, {}^{(m-2r)}C_{\frac{m-2r}{2}}\frac{\Gamma(H_0+\frac{1}{2})}{\Gamma(H_0-\frac{m-2r}{2}+\frac{3}{2})}\right) \end{aligned} . \tag{2.92}$$

This is exactly equivalent with the frequency operator obtained by using the evolution operator technique (2.81) and the renormalization group technique (2.86).

2.7 Generalized quantum anharmonic oscillator using eigenvalue approach

This straightforward operator method is based on a generalization of the harmonic oscillator algebra for the creation and annihilation boson operators. This method was first used by Fernández for the purpose of obtaining the first order correction to the frequency operator for particular values of m. Later on we have generalized his idea to obtain a generalized expression for the first order frequency operator [40]. The generalized treatment is given below.

In this approach the Hamiltonian operator H is given and we look for an operator b such that [35-41]

$$\begin{aligned} [H,b] &= -\Omega b \\ [H,\Omega] &= 0,\ \Omega^\dagger = \Omega \end{aligned} \tag{2.93}$$

Under such conditions it is possible to obtain eigenfunctions common to H and Ω:

$$\begin{aligned} H\Psi_n &= E_n\Psi_n, \\ \Omega\Psi_n &= \Omega_n\Psi_n. \end{aligned} \tag{2.94}$$

It follows from the equations (2.93 and 2.94) that

$$(E_n - E_m + \Omega_n)\langle\Psi_n|b|\Psi_m\rangle = 0. \tag{2.95}$$

Since the operator b is nonzero, there must be a pair of states Ψ_m and Ψ_n such that $\langle\Psi_m|b|\Psi_n\rangle \neq 0$; therefore,

$$\Omega_n = E_m - E_{n.} \tag{2.96}$$

The time evolution of the operator b is given by

$$b(t) = e^{itH}be^{-itH} = e^{-it\Omega}b; \tag{2.97}$$

therefore,

$$\langle\Psi_n|b(t)|\Psi_m\rangle = e^{-it\Omega_n}\langle\Psi_n|b|\Psi_m\rangle = e^{it(E_n-E_m)}\langle\Psi_n|b|\Psi_m\rangle. \tag{2.98}$$

If we can write an operator O in terms of b and $b^\dagger$ as $O = O(b, b^\dagger)$ then we have

$$e^{itH}Oe^{-itH} = O(e^{-it\Omega}b, b^\dagger e^{it\Omega}) \tag{2.99}$$

If we rewrite equation (2.97) as $e^{it(H+\Omega)}b = be^{itH}$ and expand both sides in Taylor series about $t = 0$ we obtain $(H + \Omega)^k b = bH^k$. Consequently, for any operator function $F(H)$ that can be expressed as an H-power series we have [41]

$$bF(H) = F(H + \Omega)b, \tag{2.100}$$

which is consistent with equation (2.96). In order to generalize this result we can define

$$b^kF(H) = F(k,H)b^k,\ F(0,H) = F(H). \tag{2.101}$$

A straightforward calculation shows that

$$F(k,H) = F(k-1, H + \Omega(H)). \tag{2.102}$$

It is not difficult to prove that $b^\dagger b$ is a constant of the motion

$$[H, b^\dagger b] = b^\dagger \Omega b - b^\dagger \Omega b = 0 \tag{2.103}$$

but that $bb^\dagger$ is not

$$[H, bb^\dagger] = -\Omega bb^\dagger + bb^\dagger \Omega = [\Omega(H + \Omega(H) - \Omega(H+\Omega)) - \Omega(H)]bb^\dagger, \tag{2.104}$$

unless $\Omega(H+\Omega) = \Omega(H)$.

In general, it is not possible to solve equations (2.93) except for some trivial models. One can, however, obtain approximate solutions by means of perturbation theory. If we write $H = H_0 + \lambda H'$ and expand

$$\Omega = \sum_{j=0}^{\infty} \Omega_j \lambda^j, \; b = \sum_{j=0}^{\infty} b_j \lambda^j \tag{2.105}$$

then we obtain the coefficients Ω_j and b_j from [35,36]

$$\begin{array}{lcl} [H_0, \Omega_j] & = & [\Omega_{j-1}, H'], \\ [H_0, b_j] + \Omega_0 b_j & = & [b_{j-1}, H'] - \sum_{k=1}^{j} \Omega_k b_{j-k} \end{array} . \tag{2.106}$$

For concreteness and simplicity we consider dimensionless anharmonic oscillators (1.29). In two recent papers Fernández [35, 36] obtained $\Omega_1(H_0)$ for several values of m, and $\Omega_1(H_0)$ and $\Omega_2(a, a^\dagger)$ for $m = 4$. These operators enable us to derive energy differences in the form of λ-power series:

$$\Omega_n = \sum_{j=0}^{\infty} \Omega_{n,j} \lambda^j, \; E_n = \sum_{j=0}^{\infty} E_{n,j} \lambda^j \tag{2.107}$$

In the present case $\Omega_0 = 1$ so that Ω_1 commutes with H_0. For sufficiently small λ we have $b = a + \mathcal{O}(\lambda)$ and we may choose $m = n+1$ in equation (2.96), so that

$$\Omega_{n,j} = E_{n+1,j} - E_{n,j}. \tag{2.108}$$

For the first order we have:

$$\Omega_{n,1} =< n|\Omega_1|n >, \tag{2.109}$$

where $|n\rangle$ is an unperturbed state. The operator Ω_1 is diagonal and does not contribute to the correction of second order; therefore

$$\Omega_{n,2} =< n|\Omega_2|n >; . \tag{2.110}$$

All the expressions of first and second order derived by means of the method just outlined [35, 36] already satisfy equations (2.108-2.110). Fernández has obtained Ω_1 for different m in this approach. Here we establish a general form of Ω_1 valid for all integer values of m. To do so we use the fact that Ω_1 is always a function of the unperturbed Hamiltonian H_0. Now

$$\begin{array}{lcl} < n|\Omega_1|n > & = & E_{n+1} - E_n \\ & = & \frac{1}{2^{\frac{m}{2}} m} \sum_{r=0}^{(\frac{m}{2}-1)} t_{2r} \, {}^mC_{2r} \, {}^{m-2r}C_{\frac{m-2r}{2}} \, {}^nC_{\frac{m-2r}{2}-1} \, (\frac{m-2r}{2})! \\ & = & \frac{1}{2^{\frac{m}{2}} m} \sum_{r=0}^{(\frac{m}{2}-1)} t_{2r} \, {}^mC_{2r} \, {}^{m-2r}C_{\frac{m-2r}{2}} \frac{n!}{(n-\frac{m-2r}{2}+1)!(\frac{m-2r}{2}-1)!} \, (\frac{m-2r}{2})! \\ & = & < n|\frac{1}{2^{\frac{m}{2}} m} \sum_{r=0}^{(\frac{m}{2}-1)} t_{2r} \, {}^mC_{2r} \, {}^{m-2r}C_{\frac{m-2r}{2}} \frac{\Gamma(a^\dagger a+1)}{\Gamma(a^\dagger a-\frac{m-2r}{2}+2)!} \, (\frac{m-2r}{2})|n > \\ & = & < n|\frac{1}{2^{\frac{m}{2}} m} \sum_{r=0}^{(\frac{m}{2}-1)} t_{2r} \, {}^mC_{2r} \, {}^{m-2r}C_{\frac{m-2r}{2}} \frac{\Gamma(H_0+\frac{1}{2})}{\Gamma(H_0-\frac{m-2r}{2}+\frac{3}{2})!} \, (\frac{m-2r}{2})|n > \end{array} \tag{2.111}$$

Therefore we have

$$\Omega_1 = \frac{1}{2^{\frac{m}{2}}m}\sum_{r=0}^{(\frac{m}{2}-1)} t_{2r}\,{}^{m}C_{2r}\,{}^{m-2r}C_{\frac{m-2r}{2}}\frac{\Gamma(H_0+\frac{1}{2})}{\Gamma(H_0-\frac{m-2r}{2}+\frac{3}{2})!}\left(\frac{m-2r}{2}\right) \tag{2.112}$$

This coincides exactly with the first order frequency operator obtained by other techniques.

2.8 Comparison among the different approaches

We have seen that different approaches may be used to obtain the first order frequency operator and the first order position operator. But all the results obtained in these methods are not the same. In this section we have checked the equivalence of different methods and have established some relationships between apparently different but equivalent frequency operators. To begin with, let us compare the first order cases in the next subsection and higher order cases in the subsequent subsection.

2.8.1 First order case

In the preceding sections we have obtained the same frequency operator Ω from perturbation theory in the interaction picture, renormalization group, near-identity transformation, and eigen operator techniques. In these techniques we usually write

$$X(t) = A\left[\exp(-i\Omega t)a(0) + a^\dagger(0)\exp(i\Omega t)\right] \tag{2.113}$$

where $\Omega = \Omega(H_0) = 1 + \lambda\Omega_1(H_0) + \mathcal{O}(\lambda^2)$ and A is the normalization factor. However, this form of the frequency operator appears to be different from the one derived from the Taylor series method [32, 38] and multiple-scale analysis [30, 31 and 37]. In these approaches position operator $X(t)$ is usually expressed as

$$X(t) = \cos(\omega t)X(0) + X(0)\cos(\omega t) + \sin(\omega t)\dot{X}(0) + \dot{X}(0)\sin(\omega t) + \mathcal{O}(\lambda), \tag{2.114}$$

where $\omega = \omega(H_0) = 1 + \lambda\omega_1(H_0) + \mathcal{O}(\lambda^2)$. Taking into account a particular form of equation (2.100): $aF(H_0) = F(H_0+1)a,\ F(H_0)a^\dagger = a^\dagger F(H_0+1)$, we easily rewrite equation (2.114) as

$$\begin{aligned} X(t) = \quad & \sqrt{2}\cos\left(\lambda t\frac{\omega_1-\omega_1'}{2}\right)\left[\exp\left(-it - i\lambda t\frac{\omega_1+\omega_1'}{2}\right)a\right. \\ & \left. +a^\dagger\exp\left(it + i\lambda t\frac{\omega_1+\omega_1'}{2}\right)\right] + O(\lambda^2), \end{aligned} \tag{2.115}$$

where $\omega_1' = \omega_1(H_0+1)$. Upon comparing equations (2.113) and (2.115), and taking into account that $\cos\left(\lambda t\frac{\omega_1-\omega_1'}{2}\right) = 1 + \mathcal{O}(\lambda^2)$, we conclude that

$$\Omega_1(H_0) = \frac{\omega_1(H_0) + \omega_1(H_0+1)}{2}. \tag{2.116}$$

The inverse transformation

$$\omega_1(H_0) = \sum_{j=0} c_j \left.\frac{d^j\Omega_1(\xi)}{d\xi^j}\right|_{\xi=H_0},\ c_0 = 1,\ c_j = \frac{2}{j!}\left.\frac{d^j}{d\xi^j}\left(1+e^\xi\right)^{-1}\right|_{\xi=0} \tag{2.117}$$

also gives a suitable analytical expression because both Ω_1 and ω_1 are polynomial functions of H_0.

2.8.2 Corrections of higher order

The calculation of perturbation corrections of higher order and comparison of expressions coming from different methods is considerably more difficult. We can, however, draw some useful conclusions from available results. For example, Aks and Carhart [29] showed that the Hamiltonian operator $H(p(t), x(t))$, in terms of their perturbation expressions free from secular terms for $x(t)$ and $p(t)$, is diagonal and gives the correct energy through first order perturbation theory. Unfortunately, they did not verify whether their second-order results were consistent with this criterion.

Egusquiza and Valle Basagoiti obtained an expression of the form

$$\begin{aligned} a(\tau) &= a\exp\left\{\frac{-3i\tau\lambda(H_0-\frac{1}{2})}{4}+\frac{3i\tau\lambda^2\left[5\left(H_0-\frac{1}{2}\right)^2-1\right]}{64}\right\} \\ &= \exp\left\{\frac{-3i\tau\lambda(H_0+\frac{1}{2})}{4}+\frac{3i\tau\lambda^2\left[5\left(H_0+\frac{1}{2}\right)^2-1\right]}{64}\right\}a \end{aligned} \tag{2.118}$$

by means of renormalization [33]. The second-order part of their frequency operator

$$\Omega_2^{EB} = -\frac{3}{64}\left(5H_0^2+5H_0+\frac{1}{4}\right). \tag{2.119}$$

yields

$$\langle n-1|\Omega_2^{EB}|n-1\rangle = -\frac{3}{64}(5n^2-1) \tag{2.120}$$

which does not agree with the well-known result

$$E_{n,2}-E_{n-1,2} = -\frac{3}{64}(17n^2+7) \tag{2.121}$$

that one easily obtains by means of Rayleigh-Schrödinger perturbation theory [35].

By means of the method outlined in Section 2.8 Fernández obtained[36]

$$\begin{aligned} \Omega_2 &= \tfrac{3}{16}\left(\tfrac{1}{4}a^{\dagger 4}+a^{\dagger 3}a+\tfrac{3}{2}a^{\dagger 2}+a^{\dagger}a^3+\tfrac{1}{4}a^4+\tfrac{3}{2}a^2\right) \\ &- \tfrac{3}{64}\left(17a^{\dagger 2}a^2+51a^{\dagger}a+24\right) \\ &= -\tfrac{1}{64}\left(69H_0^2+51H_0-12X^4+\tfrac{153}{4}\right) \end{aligned} \tag{2.122}$$

which is consistent with equations (2.110) and (2.121). This expression shows off-diagonal terms and agrees with the perturbation expansion of Speliotopoulos's frequency operator equation [41] when $\lambda = 4\epsilon$. There seems to be a conceptual oversight in Egusquiza and Valle Basagoiti's correction of second order to the frequency operator because it does not show offdiagonal terms and does not give the correct energy difference [33]. To be more precise, the second order frequency operator obtained by them differs from all other existing results [see equation 2.10 of ref. 33] and the discrepancy is not due to a difference in operator ordering. This fact may be established from their expression for the energy difference:

$$E_n-E_{n-1} = \left(1+\frac{3\lambda}{4}n-\frac{3\lambda^2}{64}(5n^2-1)+O(\lambda^3)\right). \tag{2.123}$$

They obtained (2.123) as a consequence of equation 2.10 of ref. [33]. Equation (2.123) is equation 2.11 of ref. [33] and it differs from the energy difference obtained by simple perturbation calculation which is

$$E_n-E_{n-1} = \left(1+\frac{3\lambda}{4}n-\frac{\lambda^2}{64}(51n^2+21)+O(\lambda^3)\right). \tag{2.124}$$

Possibly the relevant secular term chosen by them (see equation 2.8 of [33]) is incomplete and this causes the discrepancy in the expression for frequency operator obtained by them and others.

By means of a completely different approach we obtain [38]

$$\omega_2 = -\frac{1}{64}\left(69H_0^2 - 12x^4 + \frac{51}{4}\right) \tag{2.125}$$

which is consistent with their recipe for the second-order energy difference [37]

$$2\left(E_{n,2} - E_{n-1,2}\right) = \langle n|\omega_2|n\rangle + \langle n-1|\omega_2|n-1\rangle. \tag{2.126}$$

Equations (2.121) and (2.126) suggest that the diagonal parts of the operators (2.122) and (2.125) are related by

$$\Omega_2^d(H_0) = \frac{\omega_2^d(H_0) + \omega_2^d(H_0+1)}{2}, \tag{2.127}$$

as one easily verifies by means of the operators themselves.

Aks and Carhart [29] forced the frequency operator to be diagonal at all perturbation orders. They proved that their results give the Rayleigh-Schrödinger perturbation series of first order where we know that the frequency operator is diagonal and correctly given by all the approaches. We have seen that from second order onwards the frequency operator is a polynomial function of H, and, therefore, it should exhibit off-diagonal terms for the unperturbed states. It is not clear to us if the results of Aks and Carhart [29] are correct at second order because they did not give the corresponding expression of the coordinate explicitly. Moreover, it is difficult to compare the results of Aks and Carhart with others because the way they split the frequency operator into the different corrections to the coordinate operator differs markedly from the other techniques discussed here.

Here we can also check that if we put $H = H_0 + \frac{\lambda}{4}x^4$ in the expression of $\lambda(H)$ obtained by Speliotopoloulos [41] and take terms up to the second order in λ (where $\lambda = \frac{\epsilon_1}{\epsilon_0}$) then we obtain the second order frequency operator obtained by Fernández [36]. So the results of Fernández and Speliotopoloulos are in exact accordance with each other. Actually according to the work of Speliotopoloulos frequency operator is a function of the total Hamiltonian $H = H_0 + \lambda V$. Now if we express the frequency operator in power series of λ as $\Omega = \Omega_0 + \lambda\Omega_1(H_0 + \lambda V) + \lambda^2\Omega_2(H_0 + \lambda V) + ...$ then we observe that the first order correction to the frequency operator (Ω_1) should be a function of H_0 only since even the lowest power of V will give terms of the order of λ^2 . So it is expected that the second order frequency operator will be a function of H_0 and V but no cross term will be present but third and higher order frequency operator will contain cross terms like $H_0^p V$. This is really in accordance with the available results and we can recall that the second order frequency operator for the quartic oscillator obtained by us is

$$\Omega_2 = -\frac{1}{64}(69H_0^2 - 12x^4 + \frac{51}{4}).$$

This matches with the argument given above.

Thus we conclude that apparent differences among perturbation corrections of first and second orders to the frequency operator derived by different alternative approaches are simply due to different arrangements and ordering of products of noncommuting quantum-mechanical observable. In some cases it is not difficult to obtain the relationships between the different forms of the frequency operator as shown above. Therefore we can use any of these solutions to investigate the quantum fluctuations of coherent light in nonlinear media.

Chapter 3

Phase fluctuations of coherent light coupled to a nonlinear medium of inversion symmetry

The quantum phase problem may be stated as: How can one write down a quantum mechanical operator corresponding to the phase of a harmonic oscillator or equivalently a single mode of electromagnetic field? The concept of phase plays a very important role in the understanding of basic physics. So people are trying to answer this question from different points of view and are debating on this issue since Dirac [42] initiated the search of a quantum mechanical operator corresponding to the phase of a harmonic oscillator in 1927. Due to this seventy-five years long search and healthy debate an extensive amount of literature on this topic is now available but a satisfactory Hermitian phase operator is still unavailable. However, the observable nature of the phase demands that the corresponding operator should be a Hermitian one. There are many good reviews on this topic [58-62]. Some of them are written in a lucid manner [58,61] and some are written from much more mathematical and technical points of view [62].

A complete description of a harmonic oscillator is possible from the idea of a phase. Again the interference and diffraction pattern of electromagnetic waves can be explained convincingly with the help of a phase concept. Thus the phase in classical physics is well understood for a long time. We do not have such resemblance of phases in quantum mechanics. But it is widely believed that the quantum phase is one of the distinguishing features between quantum and classical physics. Actually quantum phase is responsible for all quantum mechanical interference phenomena. Even the discrete eigenvalues in quantum theory can be viewed as a quantum phase condition for the Schrödinger equation [58]. The examples of quantum phase are from double slit experiment [63] to the dark resonances [64] and from Aharonov-Bohm [65] effect to Bose-Einstein condensate [66].

Quantum mechanical phases can be classified into two main parts depending upon their origin. The phase having geometrical origin is quite familiar to us and is called geometric quantum phase (e.g Berry's phase [67], Aharonov Anandan phase [68]). The other type of phase arises due to the dynamical nature of the system. Geometric phase is discussed in chapter 6. In this chapter we briefly review the quantum phase problem and study the quantum phase fluctuations of coherent light coupled to a third order nonlinear medium of inversion symmetry.

3.1 The quantum phase: Dirac approach

Dirac started his work with the assumption that the annihilation operator a can be factored out into a Hermitian function $f(N)$ of the number operator N and a unitary operator U [42]. The later one ($i.e. U$) defines the Hermitian phase operator ϕ in the following way

$$U = exp(i\phi). \tag{3.1}$$

Hence in the Dirac formalism

$$a = exp(i\phi)f(N). \tag{3.2}$$

Now the functional form of $f(N)$ may easily be obtained by using

$$N = a^\dagger a = f^2. \tag{3.3}$$

Thus the explicit expression for a is given by

$$a = exp(i\phi)N^{\frac{1}{2}} \tag{3.4}$$

where equations (3.2) and (3.3) have been used. The expression (3.4) looks like the polar decomposition of a complex number (a). The usual commutation relation between a and $a^\dagger$ is

$$aa^\dagger - a^\dagger a = 1. \tag{3.5}$$

From equations (3.4) and (3.5) we have

$$exp(i\phi)Nexp(-i\phi) - N = 1 \tag{3.6}$$

or,

$$exp(i\phi)N - Nexp(i\phi) = exp(i\phi). \tag{3.7}$$

Equation (3.7) is valid only if the following commutation relation holds good

$$[N, \phi] = i. \tag{3.8}$$

The justification of the above condition (3.8) may be simply given in the following way. If equation (3.8) is true then from the method of induction we have

$$[N, \phi^n] = in\phi^{n-1}. \tag{3.9}$$

Multiplying both sides of equation (3.9) by $\frac{i^n}{n!}$ and summing from 0 to ∞ we arrive at equation (3.7). Now the uncertainty relation corresponding to equation (3.8) is given by

$$\triangle N \triangle \phi \geq \frac{1}{2}. \tag{3.10}$$

The equations (3.8) and (3.10) have many problems. Let us see them one by one.

3.1.1 Problems with the Dirac approach

Problem-1: The uncertainty relation (3.10) allows the uncertainty in ϕ (i.e $\Delta\phi$) greater than 2π for $\triangle N < \frac{1}{4\pi}$. This is impossible and is inconsistent with the sense of a phase. This problem is known as the periodicity problem. In this context we can recall that the uncertainty relation for L_z and ϕ also has the same problem [69-70].
Problem- 2: The Hermitian nature of ϕ (as it is expected in quantum mechanics) demands that the operator U should be an unitary one. However U is not an unitary operator.
Problem 3: In 1963 Louisell [71] showed that equation (3.8) leads to confusing results when one attempts to evaluate the matrix element of ϕ in a representation in which N is diagonal. Since,

$$< l|N\phi - \phi N|n >= i < l|n >$$

so we have

$$< l|\phi|n >= -i\frac{\delta_{n,l}}{(n-l)} \tag{3.11}$$

where $N|n >= n|n >$ has been used. If the eigenvalues of N are integer then the right hand side of equation (3.11) becomes undefined. Only in the limit of large values of n and l (Correspondence limit) ϕ takes on a definite meaning. Thus ϕ has a definite meaning only in the classical limit. This is quite consistent with the classical idea of phases.

3.1.2 Why does the Hermitian phase operator not exist?

In the previous section we have seen that the Hermitian phase operator does not exist. But a question remained unanswered: Why does a Hermitian phase operator not exist? J. Sarfatt, one of the pioneer in quantum phase problem, addressed this question for the first time. He wrote a nice paper in connection with the complexity of the introduction of quantum phase [72]. In Sarfatt words [72]'...Recent work by L. Susskind, J. Glogower and J. Sarafatt shows that it is impossible to define a phase operator because of the existence of a lowest state for the number operator of the oscillator. Thus the uncertainty relation $\triangle N \triangle\phi \geq 1$ is meaningless....'

The main cause for the nonunitarity of U is that the number state matrix representation of the diagonal number operator is bound from below. So a bare phase operator does not exist. This observation provoked people to introduce geometrical functions of phase operators.

3.2 Periodic function of phase: Louisell's approach

Louisell [71] proposed a self consistent way to avoid these difficulties. Actually we can write equation (3.9) in an alternative form as

$$[N, \phi^n] = in\phi^{n-1} = i\frac{d\phi^n}{d\phi}. \tag{3.12}$$

Therefore for any polynomial function $P(\phi)$ of φ we have a commutation relation

$$[N, P(\phi)] = i\frac{dP(\phi)}{d\phi}. \tag{3.13}$$

Louisell started his work with the fundamental commutation relation (3.13) and supposing that $P(\phi)$ is a periodic function of period 2π. He considered $\sin(\phi)$ and $\cos(\phi)$ as the Hermitian operators which satisfy the following commutation relations,

$$[N, \cos(\phi)] = -i\sin(\phi) \tag{3.14}$$

and

$$[N, \cos(\phi)] = i\sin(\phi). \tag{3.15}$$

Therefore, the uncertainty relations take the following forms

$$\triangle N \triangle \cos(\phi) \geq \frac{1}{2}| < \sin(\phi) > | \tag{3.16}$$

$$\triangle N \triangle \sin(\phi) \geq \frac{1}{2}| < \cos(\phi) > |. \tag{3.17}$$

It is to be noted that the above uncertainty relations (3.16 and 3.17) are free from problem 1. Instead of a bare phase operator, Louisell defined a meaningful phase operator in terms of the periodic function of phase. But the following questions remain unanswered:

1) What are the explicit forms of sine and cosine operators?

2) Are they Hermitian?

3.3 sine and cosine operators: Susskind and Glogower approach

The questions asked in the previous subsection were properly addressed by Susskind and Glogower (SG) [73]. They reaffirmed that London's result [74, 75] (i.e U is not unitary) are indeed correct and obtained an explicit form of sine operator S and cosine operator C as

$$S = \frac{1}{2i}(E_- - E_+) = \frac{1}{2i}\left[\frac{1}{(N+1)^{\frac{1}{2}}}a - a^\dagger\frac{1}{(N+1)^{\frac{1}{2}}}\right] \tag{3.18}$$

and

$$C = \frac{1}{2}(E_- + E_+) = \frac{1}{2}\left[\frac{1}{(N+1)^{\frac{1}{2}}}a + a^\dagger\frac{1}{(N+1)^{\frac{1}{2}}}\right]. \tag{3.19}$$

It is easy to check that the operators S (3.18) and C (3.19) are Hermitian. Therefore, SG approach is fully capable of giving the answers to the questions remained unsolved after Louisell's work. Now the commutator of C and S may be calculated by using (3.19 and 3.18) and is given by

$$[C, S] = \frac{i}{2}P^0, \tag{3.20}$$

where P^0 is the projection onto the ground state. The noncommuting nature of C and S (3.20) is one of the serious drawbacks of SG approach. This is because the cosine and sine functions of a real phase should commute. Moreover, the sum of the squares of the S and C operators is given by

$$C^2 + S^2 = 1 - \frac{1}{2}P^0. \tag{3.21}$$

Therefore, in SG approach we got satisfactory answers to the previous questions. However, we ended up with two new problems which essentially go against our perception.

3.4 Measured phase operators: Pegg and Barnett approach

In 1986 Pegg and Barnett [76] proposed a Hermitian phase operator in an extended Hilbert space containing negative number states. Such a proposal seems to be unphysical. So very soon they gave up this proposal and started their new venture with a new proposal. They proposed that the infinite dimensional harmonic oscillator Hilbert space should be truncated to $(s+1)$ dimensions and expectation values of the quantum mechanical operators should be taken in this truncated Hilbert space only. After doing that one should take the limit $s \to \infty$. This prescription of working in a truncated Hilbert space was not new in Physics. Popov-Yarunin, Sinha and many others already did some work on the finite dimensional Hilbert space. But the Pegg-Barnett prescription made the problem much more user friendly and it also defined a Hermitian phase operator in the truncated Hilbert space. So this formalism got a lot of attention of the people.

Pegg and Barnett started their work by defining a state having a well defined phase [43-44] as

$$|\theta >= lim_{s\to\infty}(s+1)^{-\frac{1}{2}}\sum_{n=0}^{s}\exp(in\theta)|n> \tag{3.22}$$

where $|n>$ is the number state spanning the $(s+1)$-dimensional space. In this space they selected a reference phase state

$$|\theta_0 >= (s+1)^{-\frac{1}{2}}\sum_{n=0}^{s}\exp(in\theta_0)|n> \tag{3.23}$$

and found the subset of states $|\theta_m>$, defined by replacing θ_0 by θ_m in equation (3.23), which are orthogonal to this state. Thus we have

$$<\theta_m|\theta_0 >= (s+1)^{-1}\sum_{n=0}^{s}\exp(in(\theta_0-\theta_m)). \tag{3.24}$$

One can easily prove that the phase states $|\theta_0>$ and $|\theta_m>$ are orthogonal to each other if the following relation holds

$$\theta_m = \theta_0 + \frac{2m\pi}{s+1}\,(m=0,1,....s). \tag{3.25}$$

The state (3.23) is an over-complete one and $|\theta_m>$ forms a complete set of orthogonal basis vectors spanning the state space. Therefore, the number state $|n>$ can be expanded in terms of $|\theta_m>$ as follows

$$|n >= \sum_{m=0}^{s}|\theta_m><\theta_m|n >= (s+1)^{-\frac{1}{2}}\sum_{m=0}^{s}\exp(-in\theta_m)|\theta_m> . \tag{3.26}$$

In this approach phase and number basis states are so related that a system in a number state is equally likely to be in any state $|\theta_m>$ and a system prepared in a phase state is equally likely to be found in any number state [44]. Now the problem is to find a unitary operator $\widehat{exp}_\theta(i\phi)$ whose eigenstates are the phase states $|\theta_m>$:

$$\widehat{\exp}_\theta(i\phi)|\theta_m >= \exp(i\theta_m)|\theta_m> \tag{3.27}$$

$$\widehat{\exp}_\theta(-i\phi)|\theta_m >= \exp(-i\theta_m)|\theta_m> . \tag{3.28}$$

Here, the caret indicates that the whole expression is an operator. From equation (3.26 and 3.27) we have

$$\widehat{\exp}_\theta(i\phi)|n>=(s+1)^{-\frac{1}{2}}\sum_{m=0}^{s}\exp[-i(n-1)\theta_m]|\theta_m>=|n-1>. \tag{3.29}$$

For the vacuum state ($n=0$) the resulting state is

$$\begin{array}{c}(s+1)^{-\frac{1}{2}}\sum_m\exp(i\theta_m)|\theta_m>=(s+1)^{-\frac{1}{2}}\exp[i(s+1)\theta_0]\sum_m\exp(-is\theta_m)|\theta_m>\\ =\exp[i(s+1)\theta_0]|s>\end{array}. \tag{3.30}$$

Now from equations (3.29 and 3.30) one can see that the number state representation of $\widehat{\exp}_\theta(i\phi)$ is

$$\widehat{\exp}_\theta(i\phi)=|0><1|+|1><2|+.....+|s-1><s|+\exp[i(s+1)\theta_0]|s><0| \tag{3.31}$$

and $\widehat{\exp}_\theta(-i\phi)$ is just the Hermitian conjugate to it. Thus we have unitary operator $\widehat{\exp}_\theta(i\phi)$ and we can define cosine and sine operators in terms of that. Those operators have properties that are much more desirable and familiar for the description of phase. In particular in Pegg-Barnett (PB) formalism we find

$$[\widehat{\cos}_\theta\phi]^2+[\widehat{\sin}_\theta\phi]^2=1 \tag{3.32}$$

$$[\widehat{\cos}_\theta\phi,\widehat{\sin}_\theta\phi]=0 \tag{3.33}$$

$$<n|[\widehat{\cos}_\theta\phi]^2|n>=<n|[\widehat{\sin}_\theta\phi]^2|n>=\frac{1}{2},\forall n, \tag{3.34}$$

where

$$\widehat{\cos}_\theta\phi=\frac{1}{2}[\widehat{\exp}_\theta(i\phi)+\widehat{\exp}_\theta(-i\phi)] \tag{3.35}$$

and

$$\widehat{\sin}_\theta\phi=\frac{1}{2i}[\widehat{\exp}_\theta(i\phi)-\widehat{\exp}_\theta(-i\phi)]. \tag{3.36}$$

One of the main differences between SG formalism and PB formalism is that the action of the $exp(i\phi)$ operator on the vacuum gives different results. This is basically responsible for all the differences in operator relations we have obtained in two formalisms. From equation (3.34) it is clear that in PB formalism the definition of phase operator is consistent with the fact that the phase of the vacuum is random but the case is not so in the S -G formalism.

Though in Pegg-Barnett formalism ϕ has the desirable properties of phase but still this approach is not free from objections. In this approach one measures the physical quantities at first in an arbitrarily large but finite state space of $s+1$ dimension and then takes the limit as s tends to ∞. These two operation do not commute so their applicability is questionable [77]. But Pegg and Barnett demand that their approach is physically indistinguishable with the conventional infinite state space model [43]. Correctness of the results obtained in different approaches may be compared only by experiments. Interesting experiments on quantum phase have started coming in [78].

People, however, use both Susskind-Glogower (SG) [5,79] and Pegg- Barnett (PB) [6, 80] formalisms for the studies of phase properties and the phase fluctuations of various physical systems. For example, the phase fluctuations of coherent light interacting with a nonlinear nonabsorbing medium of inversion symmetry are reported in the recent past [5,6]. It is found that both

SG [5] and PB [6] formalisms lead to same type (qualitatively) of phase fluctuations. Keeping these facts in mind, we have chosen PB approach for the present investigation.

In chapter 1 we have seen that the interaction of a single mode of electromagnetic field with a third order nonlinear medium can be described by the Hamiltonian (1.31). The equation of motion corresponding to the Hamiltonian (1.31) is given by equation (1.32) which contains cubic nonlinearity in field operator X. The exact analytical solution of the equation (1.32) is not available. However, under RWA the model Hamiltonian reduces to an exactly solvable form (1.33). Gerry [5] and Lynch [6] used this RW approximated Hamiltonian (1.33) to study the quantum phase fluctuations. On the other hand, different techniques to obtain approximate solution of (1.32) is discussed in chapter2. Here we use first order operator solution of (1.32) obtained in chapter 2 to study the quantum phase fluctuations of coherent light in third order nonlinear medium.

3.5 Time evolution of the useful operators

The purpose of the present section is to calculate the phase fluctuations of coherent light interacting with the nonlinear medium of inversion symmetry. The corresponding Hamiltonian is given by equation (1.31) and the fluctuations are measured with respect to the initial coherent state $|\alpha>$ which is defined as the right eigenket of the annihilation operator $a(0)$ corresponding to the eigenvalue equation $a(0)|\alpha>=\alpha|\alpha>$ [56]. The eigenvalue α is in general complex and may be written as $\alpha=|\alpha|exp(i\theta)$, where θ is the phase angle of α. The modulus square of α $(i.e|\alpha|^2)$ gives the number of photons present in the field prior to the interaction. From chapter 2 we can write the time evolution of the annihilation operator $a(t)$ as

$$\begin{aligned} a(t) &= X(t)+i\dot{X}(t) \\ &= D_1a(0)+D_2a^\dagger(0) \\ &- \Big[D_3a^3(0)+D_4a^{\dagger 3}(0)+D_5a^{\dagger 2}(0)a(0)+D_6a^\dagger(0)a^2(0)\Big] \end{aligned} \tag{3.37}$$

where the parameters

$$\begin{aligned} D_1 &= \left(1-i\tfrac{3\lambda}{4}t\right)\exp(-it) \\ D_2 &= -i\tfrac{3\lambda}{4}\sin t \\ D_3 &= i\tfrac{\lambda}{4}\sin t\exp(-2it) \\ D_4 &= i\tfrac{\lambda}{8}\sin 2t\exp(it) \\ D_5 &= i\tfrac{3\lambda}{4}\sin t \\ and & \\ D_6 &= i\tfrac{3\lambda}{4}t\exp(-it). \end{aligned} \tag{3.38}$$

The time evolution of the creation operator $a^\dagger(t)$ is the Hermitian conjugate of (3.37). Thus the number operator is

$$\begin{aligned} N(t) = a^\dagger(t)a(t) &= a^\dagger(0)a(0)+\left(D_1^*D_2a^{\dagger 2}(0)+h.c\right) \\ &- \left(D_1^*D_3a^\dagger(0)a^3(0)+h.c\right)-\left(D_1^*D_4a^{\dagger 4}(0)+h.c\right) \\ &- \left(D_1^*D_5a^{\dagger 3}(0)a(0)+h.c\right)-\left(D_1^*D_6a^{\dagger 2}(0)a^2(0)+h.c\right), \end{aligned} \tag{3.39}$$

where $h.c$ stands for the Hermitian conjugate. The parameters D_1^*, D_2^*, D_3^*, D_4^*, D_5^* and D_6^* are the complex conjugate of D_1, D_2, D_3, D_4, D_5 and D_6 respectively. The equation (3.39) is used to obtain

$$
\begin{aligned}
N^2(t) &= \left(a^{\dagger 2}(0)a^2(0) + a^{\dagger}(0)a(0)\right) + \left[2D_1^* D_2 \left(a^{\dagger 3}(0)a(0) + a^{\dagger 2}(0)\right) + h.c\right] \\
&- \left[2D_1^* D_3 \left(a^{\dagger 2}(0)a^4(0) + 2a^{\dagger}(0)a^3(0)\right) + h.c\right] \\
&- \left[2D_1^* D_4 \left(a^{\dagger 5}(0)a(0) + 2a^{\dagger 4}(0)\right) + h.c\right] \\
&- \left[2D_1^* D_5 \left(a^{\dagger 4}(0)a^2(0) + 2a^{\dagger 3}(0)a(0)\right) + h.c\right] \\
&- \left[2D_1^* D_6 \left(a^{\dagger 3}(0)a^3(0) + 2a^{\dagger 2}(0)a^2(0)\right) + h.c\right].
\end{aligned} \tag{3.40}
$$

The terms beyond the linear power of λ are neglected in equation (3.40).

The exponential of phase operator E and its Hermitian conjugate $E^{\dagger}$ under the PB formalism are given by [6]

$$
\begin{aligned}
E &= \left(\overline{N} + \tfrac{1}{2}\right)^{-1/2} a(t) \\
E^{\dagger} &= \left(\overline{N} + \tfrac{1}{2}\right)^{-1/2} a^{\dagger}(t)
\end{aligned} \tag{3.41}
$$

where $\overline{N}$ is the average number of photons present in the radiation field after interaction. Using (3.39) $\bar{N}$ can be written as

$$
\overline{N} = |\alpha|^2 \left[1 + \frac{\lambda}{4}\left(2(3+2|\alpha|^2)\sin t \sin(t-2\theta) + |\alpha|^2 \sin 2t \sin 2(t-2\theta)\right)\right]. \tag{3.42}
$$

Interestingly, $\overline{N}$ depends on the coupling constant λ, phase angle θ and on the free evolution time t. Thus the number of photons are not conserved. The result is not surprising since the nonconserving energy terms are included in the model Hamiltonian (1.31). However, in earlier studies [5-6] the photon numbers were conserved.

The usual cosine and sine of the phase operator are defined in the following way

$$
\begin{aligned}
C &= \tfrac{1}{2}\left(E + E^{\dagger}\right) \\
S &= -\tfrac{i}{2}\left(E - E^{\dagger}\right).
\end{aligned} \tag{3.43}
$$

The expectation values of the operators C and S are given by

$$
\begin{aligned}
< C > &= \tfrac{1}{2}\left(\overline{N} + \tfrac{1}{2}\right)^{-1/2} [(D_1 + D_2^*)\alpha + (D_1^* + D_2)\alpha^* - (D_3 + D_4^*)\alpha^3 \\
&\quad -(D_3^* + D_4)\alpha^{*3} - (D_5 + D_6^*)|\alpha|^2\alpha^* - (D_5^* + D_6)\,|\alpha|^2\alpha] \\
< S > &= -\tfrac{i}{2}\left(\overline{N} + \tfrac{1}{2}\right)^{-1/2} [(D_1 - D_2^*)\alpha - (D_1^* - D_2)\alpha^* - (D_3 - D_4^*)\alpha^3 \\
&\quad +(D_3^* - D_4)\alpha^{*3} - (D_5 - D_6^*)|\alpha|^2\alpha^* + (D_5^* - D_6)\,|\alpha|^2\alpha]
\end{aligned} \tag{3.44}
$$

where the equations (3.37-3.39), and the equations (3.41-3.43) are used. Again, the square of the averages are

$$
\begin{aligned}
< C >^2 &= \tfrac{1}{4}\left(\overline{N} + \tfrac{1}{2}\right)^{-1} [\{(D_1^2 + 2D_1D_2^*)\alpha^2 + c.c\} + 2\{|D_1|^2 + (D_1D_2 + c.c)\}|\alpha|^2 \\
&\quad -2\left(\{(D_1D_3 + D_1D_4^*)\alpha^4 + c.c\} + \{(D_1D_3^* + D_1D_4)|\alpha|^2\alpha^{*2} + c.c\}\right. \\
&\quad \left. + \{(D_1D_5 + D_1D_6^*) + c.c\}|\alpha|^4 + \{(D_1D_5^* + D_1D_6)|\alpha|^2\alpha^2 + c.c\}\right)]
\end{aligned} \tag{3.45}
$$

$$
\begin{aligned}
< S >^2 &= -\tfrac{1}{4}\left(\overline{N}+\tfrac{1}{2}\right)^{-1}\left[\{(D_1^2-2D_1D_2^*)\alpha^2+c.c\}-2\left\{|D_1|^2-(D_1D_2+c.c)\right\}|\alpha|^2\right.\\
&\quad -2\left(\{(D_1D_3-D_1D_4^*)\alpha^4+c.c\}-\{(D_1D_3^*-D_1D_4)|\alpha|^2\alpha^{*2}+c.c\}\right.\\
&\quad \left.\left.+\left\{(D_1D_5-D_1D_6^*)+c.c\right\}|\alpha|^4-\{(D_1D_5^*-D_1D_6)|\alpha|^2\alpha^2+c.c\}\right)\right]
\end{aligned}
\tag{3.46}
$$

where $c.c$ stands for the complex conjugate. Using equations (3.39-3.46) the second order variances of C, S and N can be written as

$$
\begin{aligned}
(\Delta C)^2 &= \tfrac{1}{4}\left(\overline{N}+\tfrac{1}{2}\right)^{-1}\left[\{|D_1|^2+(D_1D_2+c.c)\}-3\left\{(D_1D_3^*+D_1D_4)\alpha^{*2}+c.c\right\}\right.\\
&- \left.2\left\{(D_1D_5+D_1D_6^*)|\alpha|^2+c.c)\right\}-\{(D_1D_5^*+D_1D_6)\alpha^2+c.c\}\right],
\end{aligned}
\tag{3.47}
$$

$$
\begin{aligned}
(\Delta S)^2 &= -\tfrac{1}{4}\left(\overline{N}+\tfrac{1}{2}\right)^{-1}\left[\{-|D_1|^2+(D_1D_2+c.c)\}+3\left\{(D_1D_3^*-D_1D_4)\alpha^{*2}+c.c\right\}\right.\\
&- \left.2\left\{(D_1D_5-D_1D_6^*)|\alpha|^2+c.c)\right\}+\{(D_1D_5^*-D_1D_6)\alpha^2+c.c\}\right],
\end{aligned}
\tag{3.48}
$$

$$
(\Delta N)^2 = |\alpha|^2\left(1+\lambda\left[(3+4|\alpha|^2)\sin t\sin(t-2\theta)+|\alpha|^2\sin 2t\sin 2(t-2\theta)\right]\right) . \tag{3.49}
$$

3.6 Phase fluctuations

The usual parameters for the purpose of calculation of the phase fluctuations (for a fixed value of λ) are defined as [5-6]

$$
U\left(\theta,t,|\alpha|^2\right)=(\Delta N)^2\left[(\Delta S)^2+(\Delta C)^2\right]\Big/\left[< S >^2+< C >^2\right] \tag{3.50}
$$

$$
S\left(\theta,t,|\alpha|^2\right)=(\Delta N)^2(\Delta S)^2 \tag{3.51}
$$

and

$$
Q\left(\theta,t,|\alpha|^2\right)=S\left(\theta,t,|\alpha|^2\right)\Big/< C >^2 \tag{3.52}
$$

Now we can, in principle, analytically calculate U, S and Q by using equations (3.42-3.49). The equations (3.50-3.52) assume the following forms,

$$
U\left(\theta,t,|\alpha|^2\right) = \tfrac{1}{2}\left[1+\tfrac{\lambda}{4}\left(6(1+2|\alpha|^2)\sin t\sin(t-\theta)+3|\alpha|^2\sin 2t\sin 2(t-2\theta)\right)\right] \tag{3.53}
$$

$$
\begin{aligned}
S\left(\theta,t,|\alpha|^2\right) &= \tfrac{|\alpha|^2}{4}\left(\overline{N}+\tfrac{1}{2}\right)^{-1}\left[1+\tfrac{\lambda}{4}\left\{6(1+2|\alpha|^2)\sin^2 t\right.\right.\\
&+ 6|\alpha|^2 t\sin 2(t-\theta)+3|\alpha|^2\sin 2\theta\sin 2t\\
&+ \left.\left.4(3+2|\alpha|^2)\sin t\sin(t-2\theta)+4|\alpha|^4\sin 2t\sin 2(t-2\theta)\right\}\right]
\end{aligned}
\tag{3.54}
$$

and

$$
\begin{aligned}
Q(\theta,t,|\alpha|^2) &= \tfrac{1}{4cos^2(t-\theta)}\left[1+\tfrac{\lambda}{4}\left\{6(1+2|\alpha|^2)\sin^2 t+4(3+4|\alpha|^2)\sin t\sin(t-2\theta)\right.\right.\\
&+ |\alpha|^2[6t\sin 2(t-2\theta)+3\sin 2\theta\sin 2t+4\sin 2t\sin 2(t-2\theta)]\}\\
&- \tfrac{\lambda}{8\cos^2(t-\theta)}\{-6t\sin 2(t-\theta)+(6+4|\alpha|^2)\sin t\sin(t-2\theta)\\
&- 6(1+|\alpha|^2)\sin^2 t-2|\alpha|^2\sin t\sin(3t-4\theta)+|\alpha|^2\sin 2t\sin 2(t-2\theta)\\
&- |\alpha|^2\sin 2t\sin 2\theta-6|\alpha|^2 t\sin 2(t-\theta)\}]\,.
\end{aligned}
\tag{3.55}
$$

Hence the equations (3.53-3.55) are our desired results. In the derivation of the equation (3.55), we assume $|\alpha|^2 \neq 0$. Now, $U_0 = \frac{1}{2}$, $S_0 = \frac{1}{4}|\alpha|^2\left(|\alpha|^2+\frac{1}{2}\right)^{-1}$ and $Q_0 = \frac{1}{4\cos^2(t-\theta)}$ are the initial (i.e $\lambda = 0$) values of U, S and Q respectively. Thus U_0, S_0 and Q_0 signify the information about the phase of the input coherent light. The suitable choice of t may cause the enhancement and reduction of all the above parameters compared to their initial values. It is to be noted that the parameters S and Q contain the secular terms proportional to t. However, it is not a serious problem since the product λt is small [32]. The equations (3.53-3.55) are good enough to have the flavor of analytical results. Now we would like to discuss two interesting special cases.

3.6.1 The vacuum field effect

The radiation field with zero photon is termed as the vacuum field. The interaction of vacuum field with the medium gives rise to some interesting quantum electrodynamic effects [52]. The purpose of this subsection is to study the effects of the vacuum field on the phase fluctuations of input coherent light. In case of vacuum field ($|\alpha|^2 = 0$) the equations (3.53) and (3.55) reduce to

$$U(\theta,t) = U_0\left(1+\frac{3\lambda}{2}\sin t\sin(t-\theta)\right) \tag{3.56}$$

$$
\begin{aligned}
Q(\theta,t) &= Q_0\left[1+\tfrac{3\lambda}{2}\sin t\{\sin t+2\sin(t-2\theta)\}-\tfrac{3\lambda}{4\cos^2(t-\theta)}\right.\\
&\times \left.\left\{-t\sin 2(t-\theta)+\sin t\sin(t-2\theta)-\sin^2 t\right\}\right]
\end{aligned}
\tag{3.57}
$$

where $|\alpha|^2 \to 0$ is used to derive (3.57). Interestingly, the vacuum field itself couples with the medium and gives rise to the condition $\lambda \neq 0$. Thus the phase fluctuations for vacuum field are of purely quantum electrodynamic in nature. Now for $\theta = 0$, the parameters U and Q are enhanced compared to U_0 and Q_0 respectively. The corresponding maximum fluctuation of U is obtained if t is an odd multiple of $\pi/2$. The value of Q is infinite if t becomes an odd multiple of $\pi/2$. For $\theta \neq 0$, the parameters U and Q may be reduced or enhanced by the suitable choices of t. It is to be noted that the parameter U is 0.5 and is independent of θ in the earlier occasions [5-6]. In case of the present work, however, the parameter U depends on θ and t. The parameter S is identically zero and coincides exactly with the earlier results [5-6].

3.6.2 Phase of the input coherent light $\theta = \frac{\pi}{4}$

The equations (3.53-3.55) are still complicated even for $\theta = \pi/4$. A further simplification is made with the choice $t = \pi/4$. The equation (3.53) reduces to a simple form

$$U\left(\frac{\pi}{4},\frac{\pi}{4},|\alpha|^2\right) = U_0\left(1-\frac{3\lambda|\alpha|^2}{8}\right). \tag{3.58}$$

A huge reduction of U is possible with the increase of the photon number $|\alpha|^2$. However, care should be taken about the condition of the solution during such increase. The circular nature of the trigonometric function ensures the occurrence of similar reductions for other values of t ($t = 2m\pi + \frac{\pi}{4}$ where m is an integer). Now for $t = \pi/2$, we have

$$U\left(\frac{\pi}{4},\frac{\pi}{2},|\alpha|^2\right) = U_0\left(1+\frac{3\sqrt{2}\lambda}{4}\{1+2|\alpha|^2\}\right). \tag{3.59}$$

The equation (3.58) clearly shows the enhancement of U parameter as the photon number $|\alpha|^2$ increases. Thus we conclude by noting that the parameter U may decrease (3.58) or increase (3.59) with the increase of $|\alpha|^2$ by suitable choices of t. Similarly, one can obtain the reduction and enhancement of the remaining two parameters (Q and S) by suitable manipulations of free evolution time.

In the earlier works [5-6], the parameter Q is found to decrease with the increase of photon number till a minimum is reached. Subsequent increase of $|\alpha|^2$ causes the increase of Q. However, the parameters U and S are always enhanced compared to their initial values as $|\alpha|^2$ increases. Hence, the present results are in sharp contrast with those of the earlier studies [5-6]. The above differences are attributed due to the fact that the nonconserving energy terms are included in the model Hamiltonian.

3.7 Conclusion

The phase fluctuation of coherent light interacting with a nonlinear medium of inversion symmetry is carried out by using the PB formalism. The usual parameters for this purpose are U, Q and S. These parameters (3.53-3.55) are found to depend on θ, $|\alpha|^2$ and t. It is interesting to note that the free evolution time t is absent in the earlier works [5-6]. However, the presence of t is automatic in the present calculation. It accounts for the fact that the nonconserving energy terms are present in our model Hamiltonian.

The effect of vacuum field on the phase fluctuation parameters are expressed in closed analytical forms. It is found that the enhancement of U and Q are possible when the phase angle $\theta = 0$. However, for $\theta \neq 0$, both reduction and enhancement of U and Q are possible by suitable choices of t. The observed results are in sharp contrast with those of the earlier studies [5-6]. The parameter S reduces to zero for vacuum field and agrees exactly with those of the earlier works [5-6].

Apart from the general expressions for U, Q and S, we also made a qualitative comparison between the present results and the results obtained earlier for the identical physical system. For $\theta = \pi/4$, we obtain the reduction and enhancement of phase parameters (U, Q and S) by suitable manipulation of free evolution time t. Those results are in sharp contrast with the results already obtained for the same physical system [5-6]. Clearly, the free evolution time t makes the differences.

Of late the preparation of quantum states have been reported by several laboratories [81-83]. These production of quantum states have opened up the possibilities of experimental studies on quantum phase and hence the verification of the present results.

Chapter 4

Squeezing of coherent light coupled to a third order nonlinear medium

According to the Heisenberg's uncertainty principle we can't measure position (X) and momentum ($\dot{X}$) of a particle with greater accuracy than

$$\triangle X \triangle \dot{X} \geq \frac{1}{2} \tag{4.1}$$

where uncertainty $(\triangle C)$ in any arbitrary variable C is defined as $(\triangle C) = (< C^2 > - < C >^2)^{\frac{1}{2}}$. Although, there is no position operator for photon, fundamental canonical variables for light (component of electric field and magnetic field) satisfy an uncertainty relation of the form (4.1) and an electromagnetic field is said to be electrically squeezed field if $\triangle X$ is less than the vacuum field uncertainties in the quadrature (i.e. $(\triangle X)^2 < \frac{1}{2}$). Correspondingly a magnetically squeezed field is one for which $(\triangle \dot{X})^2 < \frac{1}{2}$. Here we are not going to discuss the phenomenon of squeezing in detail. So for a review on squeezed states, one can see the references [84,85]. In this chapter we want to examine the possibilities of observing squeezing in a physical situation in which a single mode of the quantized electromagnetic field, initially prepared in the coherent state (coherent light), is allowed to interact with a nonlinear nonabsorbing medium of inversion symmetry.

4.1 Application of quantum quartic oscillator: The squeezed states.

We have already shown that for the lowest order of nonlinearity (i.e cubic nonlinearity), corresponding Hamiltonian of the system is given by the Hamiltonian of a quartic anharmonic oscillator (1.31). Equation of motion of the corresponding oscillator is (1.32) and the second order operator solution of (1.32) is simply given by (2.25).

Let us now express the quadrature operator X obtained in (2.25) in terms of the initial $(i.e\, t = 0)$ annihilation and creation operators. It is given by

$$\begin{aligned} X(t) &= \Big[E_1 a(0) + E_2 a^3(0) + E_3 a^{\dagger 2}(0) a(0) + E_4 a^5(0) \\ &+ E_5 a^{\dagger}(0) a^4(0) + E_6 a^{\dagger 2}(0) a^3(0)\Big] + h.c \end{aligned} \tag{4.2}$$

where $h.c$ stands for the Hermitian conjugate. The remaining quadrature operator $\dot{X}(t)$ may simply be obtained by taking the time derivative of (4.2). The parameters E_i $(i = 1, 2, 3, 4, 5$ and

$6)$ are complex and are given by

$$\begin{aligned} E_1 &= \tfrac{1}{\sqrt{2}}\Big[\cos t - i\sin t - \tfrac{3\lambda}{4}\left\{t\sin t + i(t\cos t - \sin t)\right\} \\ &+ \tfrac{\lambda^2}{512}\{(468t\sin t - 63\cos t + 63\cos 3t - 216t^2\cos t) \\ &+ i(1188t\cos t - 1053\sin t - 45\sin 3t + 216t^2\sin t)\}\Big] \end{aligned} \tag{4.3}$$

$$\begin{aligned} E_2 &= \tfrac{1}{\sqrt{2}}\Big[-\tfrac{\lambda}{16}\left\{(\cos t - \cos 3t) + i(\sin 3t - 3\sin t)\right\} \\ &+ \tfrac{\lambda^2}{512}\{(156\cos t - 192t\sin t - 156\cos 3t - 144t\sin 3t) \\ &+ i(156\sin 3t - 324\sin t - 156t\cos 3t)\}\Big] \end{aligned} \tag{4.4}$$

$$\begin{aligned} E_3 &= \tfrac{1}{\sqrt{2}}\Big[-\tfrac{3\lambda}{4}\left\{t\sin t - i(t\cos t - \sin t)\right\} \\ &+ \tfrac{\lambda^2}{512}\{(936t\sin t - 126\cos t + 126\cos 3t - 432t^2\cos t) \\ &+ i(2376t\cos t - 2106\sin t - 90\sin 3t + 432t^2\sin t)\}\Big] \end{aligned} \tag{4.5}$$

$$\begin{aligned} E_4 &= \tfrac{\lambda^2}{256\sqrt{2}}\left[(5\cos t - 12t\sin t - 6\cos 3t + \cos 5t)\right. \\ &+ \left. i(6\sin 3t - \sin t - 12t\cos t - \sin 5t)\right] \end{aligned} \tag{4.6}$$

$$\begin{aligned} E_5 &= \tfrac{\lambda^2}{256\sqrt{2}}\left[(39\cos t - 48t\sin t - 39\cos 3t - 36t\sin 3t)\right. \\ &+ \left. i(39\sin 3t - 81\sin t - 36t\cos 3t)\right] \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} E_6 &= \tfrac{\lambda^2}{256\sqrt{2}}\left[(156t\sin t - 21\cos t + 21\cos 3t - 72t^2\cos t)\right. \\ &+ \left. i(396t\cos t - 351\sin t - 15\sin 3t + 72t^2\sin t)\right]\ . \end{aligned} \tag{4.8}$$

Initially prepared coherent state obeys the eigenvalue equation $a(0)|\alpha>=\alpha|\alpha>$, where α is in general complex and the parameter α is given by $\alpha = |\alpha|\exp(i\theta)$ as before. The number of photons present in the radiation field (prior to the interaction) is $N_0 = |\alpha|^2$. For $\theta = 0$ and $\theta = \pi/2$, the corresponding situations are called, in phase (electromagnetic field is in phase with the electric field) condition and out of phase condition respectively. Using (4.2) the second order variance (hereafter we shall call it as variance) of the X quadrature is calculated with respect to the initially prepared coherent state and is given by

$$\begin{aligned} (\Delta X)^2 &= <\alpha|X^2|\alpha> - <\alpha|X|\alpha>^2 \\ &= [(|E_1|^2 + |E_2|^2(9|\alpha|^4 + 18|\alpha|^2 + 6) + |E_3|^2(5|\alpha|^4 + 2|\alpha|^2) \\ &+ (2E_3^2|\alpha|^2\alpha^{*2} + 2E_1E_3|\alpha|^2 + E_1E_3^*\alpha^2 + 3E_1E_2^*\alpha^{*2} \\ &+ 6E_2E_3(|\alpha|^2\alpha^2 + \alpha^2) + 3E_2E_3^*\alpha^4 + E_1E_5\alpha^4 + 2E_1E_6|\alpha|^2\alpha^2 \\ &+ 3E_1E_6^*|\alpha|^4 + 4E_1E_5^*|\alpha|^2\alpha^{*2} + 5E_1E_4^*\alpha^{*4} + c.c)] \end{aligned} \tag{4.9}$$

where the terms beyond λ^2 are neglected. In a similar manner, the variance for other quadrature $\dot{X}(t)$ may be obtained. In absence of the interaction (i.e $\lambda = 0$), the variance (4.9) reduces to $\frac{1}{2}$. The squeezing in absolute sense of X quadrature is obtained if the value of $(\Delta X)^2 = 0$. We note that the squeezing of one quadrature automatically prohibits the squeezing of the remaining one. The variance (4.9) is extremely complicated and deserves more investigation for the discussions of squeezing effects of the input coherent light. It is not our purpose to go through such detail discussions of the squeezing arising out of the expression (4.9). Rather, we would prefer to discuss few special cases of physical interests.

It is true that both the solutions (2.25) and (2.26) are equally good for the purpose of getting $(\Delta X)^2$. There are good reasons for using the solution (2.25) instead of the solution (2.26) though the previous one contains secular terms. Because of the complex nature of the operators in solution (2.26), it is hard to calculate $(\Delta X)^2$. In addition to this, the present work takes care the solution of the quartic AHO up to the second order in λ. Both the solutions (2.25) and (2.26) are identical as long as the second order solution is concerned. The removal of the secular terms from the calculated $(\Delta X)^2$, if required, may be done by using the same Tucking-in technique.

4.1.1 Vacuum field

In case of vacuum field $(i.e\, |\alpha|^2 = 0)$, equation (4.9) reduces to an extremely simple form

$$\begin{aligned}(\Delta X)^2_{vac} &= |E_1|^2 + 6|E_2|^2 \\ &= \tfrac{1}{2} - \tfrac{3\lambda}{4}\sin^2 t + \left[\tfrac{3\lambda^2}{512}(201 - 24t^2 - 208\cos 2t + 7\cos 4t - 168t\sin 2t)\right].\end{aligned} \tag{4.10}$$

If we drop the terms involving λ^2 present in (4.10) then the corresponding variance coincides exactly with the earlier result [86]. Let us now discuss two special cases of particular interests. For $t = n\pi$, the variance (4.10) reduces to

$$(\triangle X)^2_{vac} = \frac{1}{2} - \frac{9\lambda^2}{64}n^2\pi^2. \tag{4.11}$$

Equation (4.11) depends on λ^2 only and we observe that the squeezing of the vacuum field is possible for $t = n\pi$. Note that a first order calculation can not predict the squeezing effect for the same conditions. Hence, a second order solution is not only useful but also essential for the present study. For $t = \frac{\pi}{2}$, the variance (4.10) reduces to

$$(\Delta X)^2_{vac} = \frac{1}{2} - \frac{3\lambda}{4} + \frac{3\lambda^2}{256}(208 - 3\pi^2). \tag{4.12}$$

For small coupling, the first order term $(i.e\ \propto \lambda)$ in equation (4.12) dominates over the second order term $(i.e\ \propto \lambda^2)$ and hence the squeezing effects are controlled accordingly. Clearly, the monotonic increase of the squeezing effects due to the first order term is arrested by the presence of the second order one. Thus the second order term is responsible for the reduction of the squeezing effect for large coupling.

Note that the so-called tucking-in technique is also applicable to equation (4.10). Hence, the secular terms can be summed up for all orders of λ. The corresponding expression for $(\Delta X)^2_{vac}$ may be expressed as

$$\begin{aligned}(\Delta X)^2_{vac} &= \tfrac{1}{2}\cos 2t - \tfrac{3\lambda}{4}\sin^2 t + \tfrac{3\lambda^2}{512}(201 - 208\cos 2t + 7\cos 4t) \\ &+ \tfrac{1}{2}(\cos\tfrac{3\lambda t}{4} - \cos 2t(1 - \tfrac{63\lambda^2}{64})).\end{aligned} \tag{4.13}$$

4.1.2 In phase with the electric field

We assume that the input radiation field is in phase with that of the electric field. In that case $\theta = 0$ and α is real. The corresponding variance reduces to the following form

$$\begin{aligned}(\Delta X)^2 &= (\Delta X)^2_{vac} + \tfrac{3|\alpha|^2}{4}\left[-\lambda\left(1 - \cos 2t + t\sin 2t\right)\right. \\ &+ \left.\tfrac{3\lambda^2}{64}\left(11 + 24t^2 - 32\cos 2t + 21\cos 4t + 71t\sin 2t + t\sin 4t - 64t^2\cos 2t\right)\right] \\ &+ \tfrac{|\alpha|^4\lambda^2}{128}\left[237 + 72t^2 - 256\cos 2t + 19\cos 4t - 36t\sin 2t - 36t\sin 4t - 216t^2\cos 2t\right].\end{aligned} \tag{4.14}$$

Equation (4.14) is useful for the purpose of discussing squeezed states generated due to the interaction a coherent light with a cubic nonlinear medium. For $|\alpha|^2 = 0$, corresponding equation reduces to the equation (4.10). If we now drop the terms involving λ^2, the variance (4.14) does have exact coincidence with the earlier reported results [86]. Here we are not going to present the "Tucked-in form" of the equation (4.14). This is because, we are interested in the qualitative study of $(\Delta X)^2$ up to the second order in λ for a weakly coupled system. If we put $t = \frac{\pi}{2}$, the corresponding variance reduces to

$$(\Delta X)^2 = \frac{1}{2} - \frac{3\lambda}{4} + \frac{3\lambda^2}{256}(208 - 3\pi^2) + \frac{3|\alpha|^2}{4}\left(-2\lambda + \frac{3\lambda^2}{64}(64 + 22\pi^2)\right) + \frac{|\alpha|^4\lambda^2}{16}\left(64 + 9\pi^2\right). \tag{4.15}$$

It is obvious that the equation (4.15) exhibits the monotonic nature of the squeezing as long as the first order solution is concerned. In most of the experiments, the percentage of squeezing gets saturated after a certain value of the electric field and the squeezing effects are completely destroyed if the field strength is very high. Moreover, oscillatory behaviour in the performance of squeezing is observed in experiments. These experimental facts are consistent with the present work (4.15). These experimental features also reveals the fact that higher order effects are inevitable in the calculation of squeezing. Thus the present second order calculation is more useful to predict the correct behavior of squeezing phenomenon observed due to the interaction of coherent light with dielectric media.

Chapter 5

Photon bunching, antibunching and photon statistics

People tried earnestly to understand the basic nature of the radiation field and as a result of this endeavor the concept of photon came into reality [87-88]. With the due success of the photon concept, the immediate questions that come to our mind are: How do the photons come? Are they coming together (i.e cluster) or one after another (without cluster)? The answers to these questions have started coming with the remarkable experiment of Hanbury-Brown and Twiss [89]. In their experiment, they obtained the intensity correlation of an incandescent light source and they concluded that the photons come together indeed. The phenomenon in which the photons come in a cluster is termed as photon bunching. On the other hand, the probability of detecting a coincident pair of photons from an antibunched field is less than that from a fully coherent light field with a random Poisson distribution of photons.

The explanations for photon bunching may be given in terms of the wave nature of the light in addition to the particle (quantum) nature. However, antibunching of photon can only be explained in terms of the particle concept of the radiation field. For this reason, the photon antibunching is regarded as a fully quantum mechanical phenomenon without any classical analogue. There is a natural tendency that the photons are bunched [3, 89]. However, there are several predictions (models) which go against this tendency and exhibit the photon antibunching. For example, various coherent states [90] and the degenerate parametric amplifier [91] were predicted as possible sources of getting antibunched light. In addition to this, the resonance fluorescence from a two level atom [92], the two photon Dick model [93] and the coherent light interacting with a two-photon medium [94] are possible sources for producing antibunched light. The fluorescence spectrum of a single two level atom shows the antibunching and the squeezing as well [92]. There are several examples where the antibunched light are produced in the laboratory [95-106]. In most of these experiments [97-101], the resonance fluorescence from a small number of ions [97], atoms or molecules [98-101] are studied to obtain the antibunching effects. The basic physics behind these experiments are clear. The atoms (or molecules) emit radiation and go to a ground state from where no subsequent radiation is possible. Hence the emitted photons are antibunched. The resonance fluorescence field from many-atom source is not suitable for antibunching of photons since the photons emitted are highly uncorrelated. However, a suitable phase matching condition similar to that of four-wave mixing leads to the photon antibunching even in the resonance fluorescence of a multi atomic system [102].

With the advent of different type of radiation sources, studies on the quantum statistical properties (QSP) of the radiation field are increased considerably. The QSP of the radiation field are usually studied with the help of second order correlation function for zero time delay [90, 107]

$$g^2(0) = \frac{< a^\dagger a^\dagger a a >}{< a^\dagger a >^2}. \tag{5.1}$$

Equation (5.1) can also be expressed in terms of $< N >$ and $(\triangle N)^2$ as

$$g^2(0) = 1 + \frac{(\triangle N)^2 - < N >}{< N >^2} \tag{5.2}$$

where $< N >$ is the average number of photons present in the radiation field and the parameter $(\Delta N)^2 =< N^2 > - < N >^2$ is the second order variance in photon number.

The conditions for bunching and antibunching of photons are given in the table 5.1

$g^2(0) > 1$	photons are bunched
$g^2(0) = 1$	photons are coherent
$g^2(0) < 1$	photons are antibunched

Table 5.1

One more useful parameter in the context of QSP of the radiation field is the Mandel Q-parameter [90, 107] which is defined as

$$Q = \frac{(\triangle N)^2 - < N >}{< N >} =< N > \left(g^2(0) - 1\right). \tag{5.3}$$

Statistical distribution of photons are related to the Q parameter as

value of Q	Photon Number Distribution
$Q < 0$	sub-Poissonian
$Q = 0$	Poissonian
$Q > 0$	super-Poissonian

Table 5.2

Now if we define a quantity

$$d = \left((\triangle N)^2 - < N >\right), \tag{5.4}$$

then the sign of d will essentially determine the quantum statistical properties of the radiation field. From the definition of d (5.4), Table 5.1 and Table 5.2 we can construct new conditions as

value of d	bunching/ anti-bunching	Photon Number Distribution
0	coherent	Poissonian
< 0	anti-bunched	sub-Poissonian
> 0	bunched	super-Poissonian

Table 5.3

The purpose of the present chapter is to study the photon number distribution (PND) and to investigate the possibilities of getting bunching and antibunching of photons in the interaction of a coherent light with a third order nonlinear medium of inversion symmetry. We are specially interested to keep the off-diagonal terms in the model Hamiltonian and to study the consequences of their inclusion in connection with the study of nonclassical effects. The effects of input vacuum field on photon bunching and antibunching are also taken care of.

5.1 The photon bunching and photon antibunching

In this chapter number of photons present in the radiation field prior to the interaction will be denoted by nonnegative integer $N_0 = |\alpha|^2$. Therefore, equation (3.42) may be used to write the average number of photons present in the radiation field (initially prepared in coherent state) after interaction as

$$\begin{aligned} < \alpha|a^\dagger a|\alpha > = < N > &= N_0 \left[1 + \tfrac{\lambda}{4}\left\{2(3+2N_0)\sin t \sin(t-2\theta)\right.\right. \\ &+ \left.\left. N_0 \sin 2t \sin 2(t-2\theta)\right\}\right]. \end{aligned} \tag{5.5}$$

Thus the number of post interacted photons (i.e $< N >$) is not conserved and is phase sensitive. It is not at all surprising since the Hamiltonian (1.31) contains the terms which are nonconserving in nature. A considerable amount of discussions about the inclusion and impacts of the photon number nonconserving terms in the QAHO model are available in chapter 1 and chapter 3. Now the variance $(\Delta N)^2 = < \alpha|N^2|\alpha > - < \alpha|N|\alpha >^2$ is given by (see 3.49)

$$(\triangle N)^2 = N_0 \left[1 + \lambda\left((3+4N_0)\sin t \sin(t-2\theta) + N_0 \sin 2t \sin 2(t-2\theta)\right)\right]. \tag{5.6}$$

Hence we have,

$$d = \frac{3\lambda N_0}{4}\left[(2+4N_0)\sin t \sin(t-2\theta) + N_0 \sin 2t \sin 2(t-2\theta)\right]. \tag{5.7}$$

The sign of the parameter d controls the phenomena of bunching and antibunching of post-interacted photons. However, the oscillatory nature of d ensures the revival and collapse of bunching and antibunching effects. The revival of bunching and antibunching was obtained in earlier occasion too [94]. Now we investigate how the sign of d is controlled by the phases. For $\theta = 0$, d is always positive and hence the photons are always bunched. Corresponding PND is super-Poissonian. For $\theta = \frac{\pi}{4}(\frac{3\pi}{4})$, the value of d is negative (positive) if $0 < \sin 2t < 1$ and is positive (negative) if $-1 < \sin 2t < 0$. Thus the photons are antibunched and bunched respectively for the former (later) and later (former) conditions. These statements are further corroborated by other values of phases. In the equations (5.5) and (5.6), the terms beyond linear power of λ are neglected. This is because of two reasons. Firstly, the solution available for the equation (1.32) is simple as long as we are interested up to the linear power in λ. The second reason is that we are interested about the qualitative studies of nonclassical properties of the radiation field and hence, the solution linear in λ is sufficient. It is tedious, however possible to obtain the correction up to the second order in λ for d. The second order correlation function (5.2) for zero time delay depends on the photon number N_0, coupling strength (λ), free evolution time t and on the phase angle θ. Values of t may be controlled by suitable manipulation of the length of the interaction region.

For $\theta = t/2$ we obtain, $< N >= N_0$ and $(\Delta N)^2 = N_0$. Therefore, $d = 0$ and hence the input coherent state remains unchanged during its passage through the nonlinear medium. The situation may be viewed as the propagation of coherent beam without affecting the propagating medium. The particular value of $t\,(=2\theta)$ causes a destructive interference in the absorption profile and as a result of that we observe a phenomenon which has strong resemblance with the self induced transparency (SIT) [108]. Here we can also mention one more important effect which is the outcome of the phase of the input coherent light. For $\theta = \frac{\pi}{4}$ and $-1 < \sin 2t < 0$ we obtain the antibunching of the emitted (post-interacted) photons.

5.2 The effect of vacuum field:

Now, we give the flavor of analytical results for input vacuum field. The quantized electromagnetic field with no photons are termed as vacuum field. The vacuum field is responsible for various interesting physical phenomena. For example, the vacuum field causes Lamb shift, Kasimir effects and Spontaneous emission [52]. Note that all these phenomena are purely quantum electrodynamic in nature and do not have any classical analogue. The effects of vacuum field interacting with a cubic nonlinear medium are studied to obtain the squeezing [86] and phase fluctuations [109] of input coherent light. It was found that these nonlinear effects are the outcome of inclusion of photon number nonconserving terms. Therefore, it is quite natural to explore the effects of vacuum field on the bunching and antibunching of photons. In case of input vacuum field $(i.e\, N_0 = 0)$, $< N >$ and $(\triangle N)^2$ are identically zero and hence the numerator of the equation (5.2) is zero $(i.e.\, d = 0)$. Hence, $g^2(0)$ is indeterminate. However, the denominator $< N >^2 \rightarrow 0$ faster compare to the numerator. Thus the value of $g^2(0) \gg 1$, if the sign of $d\,(i.e\, d \rightarrow 0^+)$ is positive. Therefore the photons are strongly bunched for vacuum field. It is already established that the vacuum field produces the squeezing effect [86]. Hence the retainment of the non conserving energy terms in the model Hamiltonian (1.31) leads to photon bunching and squeezing effects of the vacuum fields. Of course, the two photon coherent state with zero photon (i.e squeezed vacuum state) gives rise to the photon bunching along with the super-Poissonian photon statistics [90,107]. However the explanation of photon bunching for vacuum field were not given in these earlier publications. The situation is interesting, since, the photons are bunched although no photons are present in the radiation field! The result does not have classical analogue and is a consequence of pure quantum electrodynamic effect. The vacuum field interacts with the medium and produces photons which are bunched. The creation of photons through nonlinear interactions is ensured with the incorporation of the photon number nonconserving energy terms in the model Hamiltonian (1.31).

Interestingly, the suitable choice of phases (for example $\theta = \pi/4$ and $0 < t < \pi/2$) leads to negative $d\,(i.e\, d \rightarrow 0^-)$ and hence the photon antibunching. Thus, the antibunched photons may be produced through the nonlinear interaction between the medium and the vacuum field.

For $N_0 = 0$ the Q parameter (5.3) has indeterminate form $(\frac{0}{0})$. However, in the limit $N_0 \rightarrow 0$, the corresponding value of Q is found to be

$$Q = \frac{\frac{3\lambda}{2}\sin t \sin(t-2\theta)}{1+\frac{3\lambda}{2}\sin t \sin(t-2\theta)}. \tag{5.8}$$

Now for $\theta = 0$ and $t \neq n\pi$, corresponding PND is super-Poissonian. Interestingly, for $\theta = \pi/4$, the corresponding PND is sub-Poissonian if $0 < t < \pi/2$ and super-Poissonian if $\pi/2 < t < \pi$ respectively. Hence, the sub- and super-Poissonian distribution of the photons are obtained periodically. For $\theta = \pi/2$, the corresponding PND is sub-Poissonian for all t.

5.3 Conclusion:

We obtain the simultaneous appearance of photon bunching (classical phenomenon) and the sub-Poissonian photon statistics (nonclassical phenomenon). In other words classical phenomena may appear simultaneously with the nonclassical phenomena. These results are consistent with the previous observations [97,110]. The effects of the phase of the input coherent light on $g^2(0)$ are pointed out clearly. We obtain the bunching and antibunching of the vacuum field for $\theta \neq 0$. It is found that the vacuum field generates photons which may be bunched or antibunched.

We conclude the present chapter with the following observations.

1. As a result of the interaction between a single mode of coherent light and a nonlinear medium classical and nonclassical phenomena may appear simultaneously. There is no reason to believe that the appearance of one of the classical (nonclassical) phenomena warrants the presence of the others.

2. We report the sub-Poissonian PND and the antibunching of photons for input coherent light and vacuum field.

3. The vacuum field interacting with the nonlinear medium produces photons through nonlinear interaction.

Chapter 6

Application of the m-th anharmonic oscillator: Interaction of coherent light with an $(m-1)$-th order nonlinear medium

The concept of phase plays a very crucial role in the understanding of basic physics. In fact, phase is responsible for all the interference phenomena we observe in classical and quantum physics. So people have studied the properties of the quantum phase from different contexts since the beginning of quantum mechanics [111]. But the interest increased considerably in the recent past when Berry published his famous paper [67]. He showed that the phase acquired by the quantum system during a cyclic evolution under the action of an adiabatically varying Hamiltonian is a sum of two parts. The first part is dynamical in nature and the second is geometric. Later on Aharonov and Anandan generalized the idea of the Berry phase and they defined a geometric phase factor for any cyclic evolution of quantum system [68]. The existence of geometric phase is found in many areas of physics. The examples are from the quantum Hall effect to the Jahn-Teller effect and from the spin orbit interaction to quantum computation [7,111,112].

In the present work, interaction of electromagnetic fields with matter is modeled by AHO. These facts provoked us to study the possibilities of observing nonadiabatic geometric phase or Aharonov-Anandan phase [68,113] for anharmonic oscillators in general. In the present chapter we will find an analytic expression in closed form for the Aharonov-Anandan phase for a generalized anharmonic oscillator. The generalized expression is then used to study a particular case of physical interest in which an intense laser beam interacts with a third order nonlinear nonabsorbing medium having inversion symmetry. In this chapter results of chapters 4 and 5 are extended to the case of $(m-1)$-th order nonlinear medium in general to study the possibilities of generating nonclassical states in an $(m-1)$-th order nonlinear medium.

6.1 Aharonov-Anandan phase

In quantum mechanics of any system, one has a complex Hilbert space $\mathcal{H}$ whose nonzero vectors represent states of the system. A vector $|\psi>$ in $\mathcal{H}$ and any complex multiple of it (i.e $\lambda|\psi>$) represent the same state in the Hilbert space $\mathcal{H}$. This arbitrariness in the representation of states can be reduced by imposing the normalization condition

$$<\psi|\psi>=1. \tag{6.1}$$

There still remains an arbitrariness of a phase factor because a normalized vector $|\psi>$ and the vectors $\alpha|\psi>$ represents the same state in the Hilbert space provided that the modulus of α being unity (i.e $|\alpha| = 1$). The collection of all vectors $\alpha|\psi>$ with $|\psi>$, a fixed normalized vector and α taking all possible complex values of modulus one is called a unit ray and is denoted by $|\widetilde{\psi}>$ [114]. More general rays are consist of collection of all vectors of the form $\lambda|\psi>$. So rays represent vectors without any arbitrariness of phase.

To understand the geometric phase let us start from a simple situation in which a state vector $|\psi>$ evolves cyclically in the Hilbert space $\mathcal{H}$ and after a complete period T returns to the same state vector with a different phase which may be written as $\alpha|\psi>= \exp(i\phi)|\psi>$. Therefore,

$$|\psi(T)>= \exp{(i\phi)}\,|\psi(0)> \tag{6.2}$$

where ϕ is the total phase which has two parts. The first part is the dynamical phase and the other part is the nonadiabatic geometric phase or the Aharonov Anandan phase (AA phase). We are interested in the later type of phase which can be found out by subtracting the dynamical phase part from the total phase.

The cyclic evolution of the state vector may be described by a curve C in $\mathcal{H}$ because $|\psi>$ and $\alpha|\psi>$ are two different points in $\mathcal{H}$. Since there are infinitely many possible values of α so there are infinitely many curves C in $\mathcal{H}$ which describes the same cyclic evolution. In the projective Hilbert space of rays P or in the ray representative space all these points are represented by the same ray (i.e the same point). So we will have a single closed curve $\widehat{C}$ in the ray representative space P corresponding to the infinitely many possible curves in the usual Hilbert space. As there are no arbitrariness of phase in P so in the interval $[0, T]$ we must have

$$|\widetilde{\psi}(T)>= |\widetilde{\psi}(0)>. \tag{6.3}$$

Aharonov and Anandan exploited this property of the single valuedness of the unit ray in the ray representative space P to establish the presence of a geometric phase for all cyclic evolutions. They started their work by supposing that the normalized state $|\psi(t)>\in\mathcal{H}$ evolves according to the Schrödinger equation

$$H(t)|\psi(t)>= i\hbar\left(\frac{d}{dt}|\psi(t)>\right). \tag{6.4}$$

We can always define a state vector $|\widetilde{\psi}(t)>$ in P which is equivalent to $|\psi(t)>$ as

$$|\widetilde{\psi}(t)>= \exp(-if(t))|\psi(t)>. \tag{6.5}$$

Now substituting (6.2) and (6.5) in (6.3) we obtain

$$f(T) - f(0) = \phi. \tag{6.6}$$

If we assume that the total wave function rotates by 2π radian for this T which results in a cyclic motion of every state vector of the Hilbert space $\mathcal{H}$, then we obtain another condition as

$$f(T) - f(0) = 2\pi. \tag{6.7}$$

Mere satisfaction of condition (6.7) will ensure the single valuedness of the wave function in the usual Hilbert space $\mathcal{H}$. Again from (6.4) and (6.5) we have

$$H|\psi(t)>= i\hbar\left(\frac{d}{dt}\exp(if(t)|\widetilde{\psi}(t)>\right) \tag{6.8}$$

or,

$$H|\psi(t)>= -\hbar\frac{df}{dt}\exp(if(t)|\widetilde{\psi}(t)> +i\hbar\exp(if(t)\frac{d}{dt}|\widetilde{\psi}(t)> . \tag{6.9}$$

Therefore,

$$<\psi(t)|H|\psi(t)>= -\hbar\frac{df}{dt}<\psi(t)|\exp(if(t)|\widetilde{\psi}(t)> +i\hbar<\psi(t)|\exp(if(t)\frac{d}{dt}|\widetilde{\psi}(t)> \tag{6.10}$$

or,

$$-\frac{df}{dt}<\psi(t)|\psi(t)>= \frac{1}{\hbar}<\psi(t)|H|\psi(t)> - <\widetilde{\psi}(t)|i\frac{d}{dt}|\widetilde{\psi}(t)> . \tag{6.11}$$

Since $\psi(t)$ is normalized so we have

$$-\frac{df}{dt} = \frac{1}{\hbar}<\psi(t)|H|\psi(t)> - <\widetilde{\psi}(t)|i\frac{d}{dt}|\widetilde{\psi}(t)> . \tag{6.12}$$

Now integrating equation (6.12) from 0 to T and using (6.6) we have

$$\phi = -\frac{1}{\hbar}\int_0^T <\psi(t)|H|\psi(t)> dt + \int_0^T <\widetilde{\psi}(t)|i\frac{d}{dt}|\widetilde{\psi}(t)> dt = \gamma + \beta \tag{6.13}$$

where

$$\gamma = -\frac{1}{\hbar}\int_0^T <\psi(t)|H|\psi(t)> dt \tag{6.14}$$

is proportional to the time integral of the expectation value of the Hamiltonian, it is dynamical in origin and hence called dynamical phase. Now if we subtract the dynamical phase part γ from the total phase ϕ then we get the geometric phase or the Aharonov Anandan phase

$$\beta = \int_0^T <\widetilde{\psi}(t)|i\frac{d}{dt}|\widetilde{\psi}(t)> dt. \tag{6.15}$$

It is clear that the same $|\widetilde{\psi}(t)>$ can be chosen for infinitely many possible curves C by appropriate choice of $f(t)$. This phase is geometrical in the sense that it does not depend on the Hamiltonian responsible for the evolution. Moreover it is independent of the parameterization and redefinition of phase of $|\psi(t)>$. Hence, β depends only on the evolution of the shadow of $|\psi(t)>$ in projective Hilbert space P and is a pure geometrical entity.

6.1.1 Anharmonic oscillator:

The Hamiltonian of a generalized anharmonic oscillator can be written as

$$\begin{aligned} H &= H_0 + H_I \\ &= a^\dagger a + \lambda' F(a^\dagger, a) \end{aligned} . \tag{6.16}$$

Now the total eigenstate of the unperturbed Hamiltonian H_0 is

$$|\psi(0)>= \sum_n C_n|n> \tag{6.17}$$

where the coefficients C_n's depend on the initial conditions. The total wave function of the Hamiltonian H satisfying the Schrödinger equation (6.4) is

$$|\psi(t)>= \sum_n C_n\exp\left(-iHt\right)|n>= \sum_n C_n\exp\Big(-ia^\dagger at - i\lambda' F(a^\dagger, a)t\Big)|n> . \tag{6.18}$$

For the calculation of the geometric phase, we have to chose the single-valued state $|\widetilde{\psi}(t)>$. This can be easily done by choosing a $f(t)$ which simultaneously satisfies equation (6.3) and (6.7). The simplest choice is $f(t)=\lambda' t$ and with that choice we have

$$\begin{array}{rcl} |\widetilde{\psi}(t)> & = & \exp(-if(t))|\psi(t)> \\ & = & \sum_n C_n \exp\left(-ia^\dagger at - i\lambda' t\left(1+F(a^\dagger,a)\right)\right)|n> \end{array} \tag{6.19}$$

and

$$f(T)=\lambda' T = 2\pi. \tag{6.20}$$

Now from equations (6.15, 6.19 and 6.20) a compact expression for the Aharonov Anandan phase of a generalized anharmonic oscillator is obtained as

$$\begin{array}{rcl} \beta & = & \int_0^T <\widetilde{\psi}(t)|i\frac{d}{dt}|\widetilde{\psi}(t)> dt \\ & = & \int_0^T \sum_k \sum_n <k|C_k^* C_n \exp\left(ia^\dagger at + i\lambda' t\left(1+F(a^\dagger,a)\right)\right) \\ & \times & \left[a^\dagger a + \lambda'\left(1+F(a^\dagger,a)\right)\right] \exp\left(-a^\dagger at - i\lambda' t\left(1+F(a^\dagger,a)\right)\right)|n> dt \\ & = & \int_0^T \sum_k \sum_n <k|C_k^* C_n \left[a^\dagger a + \lambda'\left(1+F(a^\dagger,a)\right)\right]|n> dt \\ & = & \sum_k \sum_n <k|C_k^* C_n \left[2\pi\left(1+\frac{a^\dagger a}{\lambda'}+F(a^\dagger,a)\right)\right]|n> \\ & = & 2\pi + \sum_n |C_n|^2\left(\frac{2\pi n}{\lambda'}\right) + 2\pi\sum_k\sum_n <k|C_k^* C_n\left(F(a^\dagger,a)\right)|n> . \end{array} \tag{6.21}$$

The quantum nature of the geometric phase is manifested through the discrete sum appearing in the expression (6.21) and the state dependence of the phase appears through the coefficients $C_k^* C_n$. Now let us think of some special situations of physical interest.

6.1.1.1 Anharmonic part is a polynomial of the number operator:

When the anharmonic part can be written as a polynomial P of number operator $a^\dagger a$, i.e when $F(a^\dagger,a)=P(a^\dagger a)$ then from (6.21) we obtain

$$\beta = \sum_n |C_n|^2\left[2\pi\left(1+\frac{n}{\lambda'}+P(n)\right)\right]. \tag{6.22}$$

One of the simplest possible polynomial of number operator is

$$P(a^\dagger a) = a^{\dagger^2} a^2 = (a^\dagger a)^2 - a^\dagger a.$$

This kind of interaction is very common in Physics and Joshi *et al* [7] have shown that a geometric phase exists for such a Hamiltonian. Actually they considered the propagation of electromagnetic radiation in a dispersive medium and under certain approximation they obtained the approximated Hamiltonian. Now from (6.22) we obtain the geometric phase for the approximated Hamiltonian of Joshi *et al* (1.33) as

$$\beta = \sum_n |C_n|^2\left[2\pi\left(1+\frac{n}{\lambda'}+(n^2-n)\right)\right]. \tag{6.23}$$

This expression for geometric phase does not coincide exactly with that of Joshi et al [7]. The term $\sum_n |C_n|^2\left[2\pi\frac{n}{\lambda'}\right]$ is absent in their calculation. A conceptual oversight is responsible for that. In their paper they have worked in the interaction picture and found out the correct and exact wave function (in the interaction picture) but they have not considered the time evolution of the differential operator $\frac{d}{dt}$ present in equation (6.15). The time evolution of the operator $\frac{d}{dt}$ in the interaction picture is $\exp(iH_0t)\frac{d}{dt}\exp(-iH_0t)$ and this is the operator one should use in equation (6.15) while working in the interaction picture. Use of the corrected operator will yield the same result.

6.1.1.2 Interaction with an (m-1)-th order nonlinear medium

Now let us think of a physical situation in which an intense electromagnetic field interacts with an $(m-1)$-th order nonlinear nonabsorbing medium. If the symmetry of the medium is chosen in such a way that the only nonlinear interaction in the medium appears due to the presence of $(m-1)$-th order susceptibility then the interaction Hamiltonian (1.29) of the system may be written in an alternative form as

$$\begin{aligned} H &= \tfrac{x^2}{2} + \tfrac{\dot{x}^2}{2} + \tfrac{\lambda}{m} x^m \\ &= a^\dagger a + \lambda'(a^\dagger + a)^m \end{aligned} \tag{6.24}$$

where $\lambda' = \frac{\lambda}{m(2)^{\frac{m}{2}}}$. The above Hamiltonian (6.24) represent a generalized anharmonic oscillator for which $F(a^\dagger, a) = (a^\dagger + a)^m$ and the geometric phase is

$$\begin{aligned} \beta &= 2\pi + \textstyle\sum_n |C_n|^2 \left(\frac{2\pi n}{\lambda'}\right) + 2\pi \sum_k \sum_n C_k^* C_n < k|(a^\dagger + a)^m|n > \\ &= 2\pi + \textstyle\sum_n |C_n|^2 \left(\frac{2\pi n}{\lambda'}\right) + 2\pi \sum_n \sum_{r=0}^{\frac{m}{2}} \sum_{p=0}^{m-2r} t_{2r}\, {}^mC_{2r}\, {}^{m-2r}C_p \\ &\times \quad C^*_{n-m+2r+2p} C_n \tfrac{[n!(n-m+2r+2p)!]^{\frac{1}{2}}}{(n-m+2r+p)!} \end{aligned} \tag{6.25}$$

where

$$t_r = \frac{(r)!}{2^{(\frac{r}{2})}(\frac{r}{2})!}. \tag{6.26}$$

Normal ordering theorems 1 and 2 are used here to derive (6.25). Now we have a closed form analytic expression (6.25) for the geometric phase β for a generalized anharmonic oscillator which depends on the photon statistics of the input radiation field. So an observation of the geometric phase may help us to conclude the nature of the input electromagnetic field. We use this generalized expression (6.25) to study a very special case.

6.1.1.3 An intense laser beam interacts with a third order nonlinear medium

Let us now consider the case where an intense electromagnetic field having Poissonian statistics ($C_n = \exp(-\frac{|\alpha|^2}{2})\frac{\alpha^n}{\sqrt{n!}}$) interacts with a third order nonlinear nonabsorbing medium. If the symmetries of the medium is chosen in such a way that the only nonlinear interaction in the medium appears due to the presence of third order susceptibility, then the Hamiltonian of the system is

$$H = \frac{x^2}{2} + \frac{\dot{x}^2}{2} + \frac{\lambda}{4} x^4 = H_0 + \lambda'(a^\dagger + a)^4 \tag{6.27}$$

where λ is the coupling constant and $\lambda' = \frac{\lambda}{16}$. With the help of equations (6.25) the geometric phase β for this physical system can be written as

$$\begin{aligned} \beta &= 2\pi + \textstyle\sum_n |C_n|^2 \left(\frac{2\pi n}{\lambda'}\right) + 2\pi \sum_n \sum_{r=0}^{2} \sum_{p=0}^{4-2r} t_{2r}\, {}^4C_{2r}\, {}^{4-2r}C_p C^*_{n-4+2r+2p} C_n \frac{[n!(n-4+2r+2p)!]^{\frac{1}{2}}}{(n-4+2r+p)!} \\ &= 8\pi + 2\pi \exp(-|\alpha|^2) \left\{ \textstyle\sum_n \frac{|\alpha|^{2n}}{n!} \left[\left(6n^2 + \left(6 + \frac{1}{\lambda'}\right) n \right) \right.\right. \\ &= \left.\left. 4|\alpha|^2 (2n+3) \cos(2\theta) + 2|\alpha|^2 \cos(4\theta) \right] \right\} \\ &= 8\pi + 2\pi \left[\left(6|\alpha|^4 + 12|\alpha|^2 + \frac{|\alpha|^2}{\lambda'} \right) + 4|\alpha|^2 \left(2|\alpha|^2 + 3\right) \cos(2\theta) + 2|\alpha|^2 \cos(4\theta) \right]. \end{aligned} \tag{6.28}$$

Here we observe that the geometric phase depends on the phase of the initial laser beam. The presence of the off-diagonal terms in the interaction Hamiltonian is manifested through the presence of θ dependent terms in the expression for the geometric phase β.

Photon Statistics	$\lvert C_n\rvert^2$
Sub-Poissonian	$\frac{N!}{(N-n)!n!}p^n(1-p)^{N-n}\ (0\leq p\leq 1)$
Poissonian	$\exp(-\lvert\alpha\rvert^2)\frac{\alpha^n}{\sqrt{n!}}$
Super-Poissonian	$\frac{(n+W-1)!}{(W-1)!n!}q^n(1-q)^W\ (0\leq q\leq 1)$

Table 6.1

6.1.2 Remarks on the results:

The geometric phase for the physical systems considered by us can be experimentally observed with the help of interferometric experiments. To be more precise we can think of an optical analogue of the Suter-Muller-Pines experiment [115]. A gedanken experiment for this purpose is also proposed by Joshi et al. [7].

We have seen that the geometric phase arising due to the interaction of intense electromagnetic field with a nonlinear medium is experimentally observable and the C_n's depend on the photon statistics of the input field (see Table 6.1). So an observation of the geometric phase can give us the photon statistics of the input electromagnetic field. Earlier we have shown that the photon statistics in a third order nonlinear nonabsorbing medium of inversion symmetry changes with the interaction time and in this chapter we will show that this fact is true for $(m-1)-th$ order nonlinear medium also. Using the output of such an interaction as the input of a geometric phase experiment our predictions on photon statistics can be verified.

The general Hamiltonian of the form (6.24) and its special case (6.27) are very common in quantum optics and we often use rotating wave approximation (RWA) to deal with these Hamiltonians. Under RWA calculations off-diagonal terms are neglected. But in the present case the presence of the off-diagonal terms in the interaction Hamiltonian is manifested through the presence of θ dependent terms in the expression for the geometric phase β. From these facts we can conclude two things firstly, it is not right to use RWA for the calculation of the geometric phase and, secondly, geometric phase can be tuned because it depends strongly on the phase of the input field which can be tuned.

6.2 Higher harmonic generation

If an input of frequency ω generates a output frequency $n\omega$ due to the nonlinear frequency mixing in a nonlinear medium, then we say that the $n-th$ harmonic generation is taken place in the medium. It is well known that the $n-th$ harmonic generation is observed in an $n-th$ order nonlinear medium. Now, a close look at the term, $\exp\left(i\Omega t(2p-m+2r-1)\right)$ in (2.80), will show that the expansion of the summation present in (2.80) will give us terms with frequency $3\Omega, 5\Omega, 7\Omega, ..., (m-1)\Omega$ when m is even and $2\Omega, 4\Omega, 6\Omega, ...(m-1)\Omega$ when m is odd. Thus we can conclude from the above discussion and (2.80) that the frequency mixing occurred in such a way that all the odd / even higher harmonics up to the order of the nonlinearity of the medium will be present in the output.

6.3 Bunching, antibunching and statistical distribution of the photons

In chapter 5 we have seen that the quantum statistical properties of radiation field can be studied with the knowledge of sign of $d = ((\triangle N)^2 - < N >)$. To study the QSP of the output of an interaction of coherent light with an $(m-1)$-th order nonlinear medium we can construct a closed form expression for d by using (2.78) and (5.4) as

$$\begin{array}{rcl} d & = & \frac{4\lambda}{m(2)^{\frac{m}{2}}} \left(\sum_r^{\frac{m}{2}} t_{2r} \, {}^m C_{2r} \sum_{p \neq \frac{m-2r}{2}}^{(m-2r)} \, {}^{(m-2r)} C_p \frac{p(p-1)}{2p-m+2r} |\alpha|^{m-2r} \sin\left[\frac{(2p-m+2r)}{2}(t-2\theta)\right] \right. \\ & \times & \left. \sin\left[\frac{(2p-m+2r)}{2} t\right] \right) \end{array} \tag{6.29}$$

where we have taken the expectation value with respect to the initial coherent state $|\alpha >= \sum_{n=0}^{\infty} \exp\left(\frac{|\alpha|^2}{2}\right) \frac{\alpha^n}{\sqrt{n!}} |n >$ with $\alpha = |\alpha| \exp(i\theta)$, having Poissonian statistics. From this expression we can see that for $m = 4$, i.e for a third order nonlinear medium having inversion symmetry, we have

$$d = \frac{3\lambda|\alpha|^2}{4} \left[2\left(2|\alpha|^2 + 1\right) \sin(t-2\theta)\sin(t) + |\alpha|^2 \sin\left(2(t-2\theta)\right)\sin(2t) \right]. \tag{6.30}$$

This is in exact accordance with our earlier result. From (6.29) we have following observations,

i) When $t = 2\theta$ then $d = 0$. Therefore, input coherent state remains coherent in the output.

ii) When $\theta = 0$ or $\theta = n\pi$, i.e. input is real then d is a sum of square terms only. So d is always positive and we have bunched photons having super-Poissonian statistics in output.

iii) For other values of phase θ of input radiation field, value of d oscillates from positive to negative, so we can observe bunched, antibunched or coherent output depending upon the interaction time t.

6.4 Squeezing

Here we will study the possibilities of electrically squeezed field. Possibilities of magnetically squeezed field may also be studied in the similar process. Now the uncertainty in electric field is

$$\begin{array}{rcl} (\triangle X)^2 & = & \frac{1}{2} \left[1 + \frac{2\lambda t}{2^{\frac{m}{2}} m} \sum_{r=0}^{\frac{m}{2}} t_{2r} \, {}^m C_{2r} \, {}^{m-2r} C_{\frac{m-2r}{2}} \left(\frac{m-2r}{2}\right)\left(\frac{m-2r}{2} - 1\right) |\alpha|^{m-2r-2} \sin\left(2(\theta - t)\right) \right. \\ & - & \frac{4\lambda t}{2^{\frac{m}{2}} m} \sum_r^{\frac{m}{2}} t_{2r} \, {}^m C_{2r} \sum_{p \neq \frac{m-2r}{2}}^{(m-2r)} \, {}^{(m-2r)} C_p \frac{p|\alpha|^{m-2r-2}}{(2p-m+2r)} \sin\left(\frac{(m-2r-2p)}{2} t\right) \\ & \times & \left. \left[(p-1)\sin\left(\frac{(m-2r-2p+2)}{2}(2\theta - t) - t\right) + (m-2r-2p)\sin\left(\frac{(m-2r-2p)}{2}(2\theta - t)\right) \right] \right]. \end{array} \tag{6.31}$$

Here we can note that the squeezed state may be generated due to the interaction of an intense laser beam with the $(m-1)$-th order nonlinear medium for particular values of θ and t. For example, for $\theta = 0$ and $t < \frac{2\pi}{m}$ we will get electrically squeezed field in the output. To compare the above expression with the existing results we put $m = 4$ and $\theta = 0$, which corresponds to the case in which an intense laser beam having zero input phase interacts with a third order nonlinear non absorbing medium of inversion symmetry and obtain,

$$(\triangle X)^2 = \frac{1}{2} - \frac{3\lambda}{4}\sin^2(t) - \frac{3\lambda|\alpha|^2}{4}\left(t\sin(2t) + 2\sin^2(t)\right). \tag{6.32}$$

This is in exact accordance with our earlier result [38]. From (6.32) we can see that the squeezing in electric field quadrature will be observed for $t < \frac{\pi}{2}$ for any arbitrary allowed values of other parameters. So a nonclassical phenomenon (squeezing) is observed for $m = 4$ and $\theta = 0$ but for the same conditions photons of the output radiation field are always bunched. Thus we can conclude, although the nonclassical phenomenon of antibunching and squeezing usually appears together but it is not essential that the two nonclassical phenomena have to appear together.

Chapter 7

Summary and concluding remarks

In the present work coherent light interacting with a nonlinear medium of inversion symmetry is modeled by a general quantum anharmonic oscillator. The quartic-anharmonic-oscillator model emerges if only the lowest order of nonlinearity (i.e third order nonlinear susceptibility) is to be present in the medium. The quartic-oscillator model has considerable importance in the study of nonlinear and quantum optical effects that arise in a nonlinear medium of inversion symmetry. For example, silica crystals constitute an inversion symmetric third order nonlinear medium and these crystals are used to construct the optical fibers. In the optical communication electromagnetic beam passes through the optical fiber and the interaction of the electromagnetic field (single mode) with the fiber can be described by the one dimensional quartic anharmonic oscillator Hamiltonian. Depending on the nature of nonlinearity in a physical problem the treatment of higher anharmonic oscillators also assumes significance. But anharmonic oscillator models are not exactly solvable in a closed analytic form. On the other hand, we need operator solutions of the equations of motion corresponding to these models in order to study the quantum fluctuations of coherent light in nonlinear media. So we have two alternatives, either we can use an approximate Hamiltonian which is exactly solvable or we can use an approximate operator solution. Approximate operator solutions of anharmonic oscillator problems (solutions in the Heisenberg approach) were not available even in the recent past presumably because the existing methods tend to introduce inordinate mathematical complications in a detailed study. Due to the unavailability of the operator solution people would use RW approximated Hamiltonians to study the quantum fluctuations of coherent light in nonlinear media. But the situation is now improved and many proposals to obtain approximate operator solutions of anharmonic oscillators have appeared. Some of these solutions are obtained as a part of the present work. For example, we have constructed a second order operator solution for quartic oscillator and have generalized the first order operator solutions available for the quartic oscillator to the m-th anharmonic oscillator. From the generalized solutions we have observed that an apparent discrepancy is present between the solutions obtained by different techniques. Then the question arise: Which solution should be used for physical applications? Therefore, we have compared different solutions and have concluded that all the correct solutions are equivalent and the apparent discrepancy whatsoever is due to the use of different ordering of the operators. These solutions are then exploited to study the possibilities of observing different nonlinear optical phenomena in a nonlinear dielectric medium. To be precise, we have studied quantum phase fluctuations of coherent light in third order inversion symmetric nonlinear medium in chapter 3. Fluctuations in phase space quadrature for the same system is studied in chapter 4 and the possibility of generating squeezed state is reported. In chapter 5 fluctuations in photon number is studied and the nonclassical phenomenon of antibunching is predicted. In chapter 6 we have generalized the results obtained for third order nonlinear medium and have studied the interaction of an intense laser beam with an $(m-1)$-th order nonlinear medium in general. Aharonov Anandan nonadiabatic geometric phase is also discussed in the context of $(m-1)$-th order nonlinear medium in general.

We have observed different interesting facts regarding quantum fluctuations of coherent light interacting with nonlinear media and now we can conclude the present work with the following observations.

1. In a third order nonlinear medium reduction and enhancement of phase fluctuation parameters ($U(\theta, t, |\alpha|^2)$, $Q(\theta, t, |\alpha|^2)$ and $S(\theta, t, |\alpha|^2)$) are possible for suitable choice of free evolution time t. This result is in sharp contrast with the earlier results [5-6].

2. Electrically squeezed electromagnetic field can be generated due to the interaction of an intense beam of coherent light with an $(m-1)$-th order nonlinear medium.

3. Antibunching of photons can be observed due to the interaction of an intense beam of coherent light with an $(m-1)$-th order nonlinear medium in general.

4. Simultaneous appearance of photon bunching (classical phenomenon) and squeezing (nonclassical phenomenon) is observed. This establishes the fact that there is no reason to believe that the appearance of one of the nonclassical phenomena warrants the presence of the others.

5. The vacuum field interacting with the nonlinear medium produces photons through nonlinear interaction. We report sub-Poissonian PND and antibunching of these photons.

6. A nonvanishing noadiabatic geometric phase appears due to the interaction coherent light with an $(m-1)$-th order nonlinear medium. The geometric phase for the physical systems considered by us can be experimentally observed with the help of interferometric experiments.

7. Under RWA calculations off-diagonal terms are neglected. But in the present case the presence of the off-diagonal terms in the interaction Hamiltonian is manifested through the presence of θ dependent terms in the expressions for the β, $(\triangle X)^2$ etcetera. Therefore special care should be taken before using RWA for the calculations related to the matter-field interaction.

8. Percentage of squeezing and geometric phase can be tuned because they strongly depend on the phase of the input coherent field which can be tuned.

7.1 Limitations and scope for future works

In most of the studies envisaged in the present work we have used either first order or second order solutions. But higher order terms might have important effects in comparatively strongly coupled systems. Higher order operator solutions are now appearing in the literature such that it is now technically possible to study the strongly coupled systems using the methods used in the present work.

In some particular cases we have used secular solutions to study the quantum fluctuations. This is valid for short interaction time. In order to have corresponding expressions valid for all times we have to remove the secular terms from the expressions. Tucking in technique may be used for the purpose.

We started working on the quantum anharmonic oscillator problem in 1998. At that time only two techniques were available to provide operator solution of the quantum quartic anharmonic oscillator. The situation has considerably improved during the last four years. Now we have sixth order solution of quartic oscillator [116][1]. So we can now address the question of convergence of the operator series. We can also extend our works on one dimensional anharmonic oscillators to two or higher dimensions. We also expect that the present work will serve as the back bone of the nonlinear dynamical studies (in Heisenberg approach) of more complex systems. For example, we can study a system of coupled oscillators, toda lattices etcetera. This kind of study may provide many new information in near future.

The other thing is that almost all the effects discussed in the present thesis are experimentally realizable and all of them have potential applications in optical communications and other areas of physics which are related to public life. We finish the present work with an expectation that it will be useful for the future development of theoretical and experimental studies of nonlinear dynamical systems.

[1] In fact we can construct solution of anharmonic oscillators (for any particular m) up to arbitrary order. The procedure is reported in our work [40] but it is not included in the present thesis.

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